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Toeplitz Products and Two-Weight Inequalities on Spaces of Vector-Valued Functions

by

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Abstract

This thesis is concerned with operators on certain vector-valued function spaces. Namely, Bergman spaces of $\mathbb{C}^n$-valued functions and $L^2(\mathbb{R}, \mathbb{C}^n, V)$, where V is a matrix weight. We will study products of Toeplitz operators on the vector Bergman space $L^2_a(\mathbb{C}^n)$. We also study various operators, including the dyadic shift and the Hilbert transform, between $L^2(\mathbb{R}, \mathbb{C}^n, V)$ and $L^2(\mathbb{R}, \mathbb{C}^n, U)$. These function spaces are generalizations of normed vector spaces of functions which take values in $\mathbb{C}$.

The thesis is split into two distinct areas of function space theory: analytic function spaces and harmonic analysis. There is, however, a common theme of matrix weights, particularly the reverse Hölder condition on matrix weights and a generalization of the A_p conditions on matrix weights for $p = 2$ and $p = \infty$.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow. Chapter 1 contains known results and relevant background material. The remaining chapters are the author's original work, except where explicitly referenced.

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Notation

- $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, i.e. $\{z \in \mathbb{C} : |z| < 1\}$.
- For a vector x in an arbitrary normed vector space $\mathcal{V}$, we will denote the norm of x by $||x||_{\mathcal{V}}$. The fact that all norms on a finite dimensional vector space are equivalent will be used regularly throughout this thesis.
- $|| \cdot ||_{\text{op}}$ will refer to the operator norm.
- Where x and y are vectors in a Hilbert space $\mathcal{H}$ we will denote the inner product of x and y in $\mathcal{H}$ by $\langle x, y \rangle_{\mathcal{H}}$.
- If I is a measurable subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, then $|I|$ will refer to the Lebesgue measure of I.
- $\langle V \rangle_I$ will refer to the average of V over the subset I, $0 < |I| < \infty$,
$$\frac{1}{|I|} \int_I V(x) dx,$$
where V is a locally integrable scalar, vector or matrix-valued function on $\mathbb{R}$ or $\mathbb{T}$ and I is measurable a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$.
- $V(I)$ will refer to the integral of V over I,
$$\int_I V(x) dx,$$
whenever I is a measurable subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ and V is a locally integrable scalar, vector or matrix-valued function.
- L^2_a is the Bergman space on the unit disk.
- $L^2_a(\mathbb{C}^n)$ is the Bergman space of $\mathbb{C}^n$-valued analytic functions on the unit disk.
- $L^2(\Omega, V)$ is the space of measurable vector-valued functions such that, for a weight V, the following inequality holds: $\int_\Omega ||V^{\frac{1}{2}}(x) f(x)||^2_{\mathbb{C}^n} dA(x) < \infty$.

Chapter 1

Introduction

1.1 Toeplitz Operators

One of the most commonly studied classes of operators on analytic function spaces are the Toeplitz operators. There is a vast volume of literature concerned with studying properties of these operators, as well as their numerous applications. The principal analytic function space upon which these operators are defined is the Hardy space, which we will introduce here. In this thesis we are, however, concerned with another important analytic function space: the Bergman space. The reason for introducing the Hardy space is to illustrate the context and history of one of the problems this thesis focuses upon. This problem, that the first two chapters of this thesis are concerned with, involves products of Toeplitz operators on a vector Bergman space and their boundedness. As Toeplitz operators were originally defined on the Hardy space, H^2, it is therefore unsurprising that this problem of characterizing boundedness of Toeplitz products also originated on H^2. Introducing this problem in the case of the Hardy space also serves to highlight the connection between the two distinct areas of theory that concern us: products of Toeplitz operators and the two-weight problem. The Bergman space is defined in terms of the area measure, and the Hardy space is defined in terms of a radial limit on the circle.

Definition 1.1.1. The Hardy space H^2 is the vector space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

The square root of this expression defines a Hilbert space norm on H^2. More generally we have the following definition.

Definition 1.1.2. The Hardy space H^p is the vector space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

We have defined the Hardy space here as a function space where the functions have domain $\mathbb{D}$. Other domains are possible, but for the sake of simplicity and analogy we restrict ourselves to the unit disk.

We can identify H^2 with the closed $L^2(\mathbb{T})$ subspace of functions f such that $\widehat{f}(n) = 0$ for all $n < 0$, where $\widehat{f}$ is the Fourier transform of f. Firstly each function from this subspace of L^2 can be mapped to an H^2 function via the Poisson integral:

Definition 1.1.3. The *Poisson integral* of a function $f \in L^2(\mathbb{T})$ is the function $\mathcal{P}f : \mathbb{D} \to \mathbb{C}$ defined, for $z \in \mathbb{D}$, by

$$\mathcal{P}f(z) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) P_z(\theta) d\theta,$$

where P_z is the *Poisson kernel* $\frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2}$. We will sometimes denote the Poisson integral of a function f, $\mathcal{P}f$, as f^*.

The identification between these two subspaces is made explicit in the following theorem.

Theorem 1.1.4. *The set $\{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0, \text{where } n < 0\}$, where $\widehat{f}(n)$ is the nth fourier coefficitent of f, is a subspace of $L^2(\mathbb{T})$ which is isometrically isomorphic to H^2. The isometric isomorphism is explicitly constructed in the following two statements:*

- *Let f be a function in $L^2(\mathbb{T})$ such that for all $n < 0$ the Fourier coefficient $\widehat{f}(n) = 0$, then the Poisson integral of f, $\mathcal{P}f$, is in H^2 and $||f||_{L^2(\mathbb{T})} = ||\mathcal{P}f||_{H^2}$. We can also recover our function f by taking radial limits almost everywhere. In other words, $\lim_{r\to 1} \mathcal{P}f(re^{i\theta}) = f(e^{i\theta})$ exists almost everywhere and agrees with f where it exists.*
- *For each $\widetilde{f} \in H^2$, radial limits give a function $f \in L^2(\mathbb{T})$ such that $\widehat{f}(n) = 0$ for all $n < 0$ and $\mathcal{P}f = \widetilde{f}$.*

H^2 is used to denote both of these subspaces.

Proof. This is well known and can be found in [8] for example. $\square$

The orthogonal projection, P, from L^2 onto H^2 is given by the following integral:

$$Pf(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)d\theta}{1 - ze^{-i\theta}}.$$

Toeplitz operators on this space are usually defined, with a symbol $f \in L_\infty$, as follows:

Definition 1.1.5. A Toeplitz operator T_f, with symbol $f \in L_\infty$, is the bounded operator $T_f : H^2 \to H^2$ such that, for $h \in H^2$,

$$T_f h = P(fh) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)h(\theta)d\theta}{1 - ze^{-i\theta}}.$$

In fact the following theorem illustrates why Toeplitz operators are usually defined with such a symbol, see Operator Theory in Function Spaces by Kehe Zhu [43] for a proof.

Theorem 1.1.6. *If g is a measurable function on $\mathbb{T}$, then the Toeplitz operator T_g defines a bounded linear operator on H^2 if and only if $g \in L^\infty$.*

So, in the light of this theorem it might seem likely that the product of two Toeplitz operators, $T_f T_g$, be bounded for measurable functions f and g if and only if $f, g \in L_\infty$. This is, however, not the case and in the next section we discuss some of the subtleties involved in defining a Toeplitz operator with symbol $f \in L^2$, and the possibility of products of these operators being bounded while one or both of the factors is unbounded.

1.2 Toeplitz Products on the Hardy Space and Sarason's Conjecture

Let $f \in L^2$. Then for $h \in H^\infty$ the function $T_f h = P(fh)$ is well-defined. Note here that H^∞ is dense in H^2.

We use the fact that L^2 is the Hilbert space direct sum of H^2 and $\overline{H_0^2}$, where $\overline{H_0^2} = \{\overline{f} : f \in H_0^2\}$ and $H_0^2 = \{f \in H^2 : f(0) = 0\}$. Thus each $f \in L^2$ can be written as $f = g + \overline{h}$, where g and h are H^2 functions. We can then see that the Toeplitz operator T_f is equal to the sum $T_g + T_{\overline{h}}$. For a symbol $g \in H^2$ the Toeplitz operator T_g on H^∞ is just the multiplication operator $M_g k = gk$, for $k \in H^\infty$, and its range is contained in H^2. This is because the product of two analytic functions is again analytic and the product of a bounded analytic function and an H^2 function is in H^2. More generally, for a symbol $g \in H^2$, the Toeplitz operator T_g on H^2 is also just the multiplication operator $M_g k = gk$ for $k \in H^2$, however, the resulting function won't necessarily be contained in H^2. The identification of the multiplication operator and a Toeplitz operator with H^2 symbol is well defined due to the fact that, while not bounded on L^1, the projection operator will reproduce an H^1 function.

So, if we have a product of Toeplitz operators, $T_f T_g$, with symbols $f, g \in L^2$ we can write this as

$$T_f T_g = T_{f_1+\overline{f_2}} T_{g_1+\overline{g_2}} = T_{f_1} T_{f_2} + T_{f_1} T_{\overline{g_2}} + T_{\overline{f_2}} T_{g_1} + T_{\overline{f_2}} T_{\overline{g_2}}, \tag{1.2.1}$$

where $g_1, g_2, f_1, f_2 \in H^2$. Due to the fact that $T_f^* = T_{\overline{f}}$ the boundedness of a product of the form $T_{\overline{f_2}} T_{g_1}$ is equivalent to the boundedness of the product $T_{\overline{g_1}} T_{f_2}$, and the boundedness of a product of the form $T_{f_1} T_{f_2}$ is equivalent to the boundedness of $T_{\overline{f_2}} T_{\overline{f_1}}$, where the f_i and g_i are H^2 functions. Hence the boundness of $T_{f_1} T_{f_2}$ and $T_{\overline{f_2}} T_{\overline{f_1}}$ reduces to the boundendess of the multiplication operator $M_{f_1 f_2}$. Thus, the interesting operator in this sum is the operator of the form $T_f T_{\overline{g}}$ with $f, g \in H^2$, and it is operators of this form that were the subject of the now disproven Sarason's conjecture.

Sarason's conjecture came about when classes of bounded Toeplitz products, $T_f T_{\overline{g}}$ with f and g in H^2, were found but with either one or both of T_f and $T_{\overline{g}}$ being unbounded.

Conjecture 1.2.1 (Sarason). *For functions $f, g \in H^2$ the Toeplitz product $T_f T_{\overline{g}}$ is bounded if and only if $\sup_{z \in \mathbb{D}} \left(|f|^2\right)^* (z) \left(|g|^2\right)^* (z) < \infty$. Here f^* denotes the Poisson integral of f.*

Although the conjecture is usually referred to as Sarason's conjecture, Sarason was actually more cautious with his words and instead said that "it is tempting to conjecture", this can be found in [31]. His caution was well founded as the conjecture was later found to be false, [20].

In the original reference for Sarason's conjecture [31] the question of an analogous problem in the Bergman space is raised. We will proceed to introduce this problem and the analogous conjecture, which is still an open problem.

1.3 The Bergman Space and Sarason's Conjecture

Definition 1.3.1. The *Bergman space*, $L_a^p(\mathbb{D})$, is the intersection of $L^p(\mathbb{D})$ with the analytic functions on $\mathbb{D}$ with the usual identification of functions which only differ on sets of measure 0.

The norm of an L^p_a function is inherited from L^p: $||f||_{L^p_a(\mathbb{D}} = (\int_{\mathbb{D}} |f(z)|^p dA(z))^{\frac{1}{p}}$, where dA is normalized Lebesgue measure on the unit disk $\mathbb{D}$. In the case $p = 2$ this space becomes a Hilbert space with the inner product given by $\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z)$. The Bergman space L^p_a is a Banach space for $1 \leq p < \infty$.

On the Bergman space L^2_a, the Toeplitz operator with symbol $f \in L^2(\mathbb{D})$ is the densely defined operator $T_f v = P(fv)$, where P is the orthogonal projection from $L^2(\mathbb{D})$ into L^2_a and v is a bounded analytic function on $\mathbb{D}$. The Toeplitz operator is a multiplication operator composed with the orthogonal projection. The Bergman projection is explicitly given by the following integral:

$$Pf(w) = \langle f, K_w\rangle = \int_{\mathbb{D}} \frac{f(z)}{(1-\bar{z}w)^2} dA(z),$$

where $K_w(z) = \frac{1}{(1-z\overline{w})^2}$ is the reproducing kernel of the Bergman space $L^2_2(\mathbb{D})$. So, using this explicit form we can, as we did on the Hardy space, define a Toeplitz operator, on a dense subset, H^∞, of L^2_a, with symbol in L^2 rather than L^∞. We can also see that, with a symbol $f \in L^2$ and $v \in L^2_a$, the function $T_f v(w) = \int_{\mathbb{D}} \frac{f(z)v(z)}{(1-\bar{z}w)^2} dA(z)$ is well defined pointwise for each $w \in \mathbb{D}$. Before stating Sarason's conjecture about Bergman space Toeplitz operators we need to introduce the Berezin transform.

Definition 1.3.2. The *Berezin transform* of a function $f \in L^2_a$ is the function $\widetilde{f} : \mathbb{D} \to \mathbb{C}$ given by

$$\widetilde{f}(w) = \int_{\mathbb{D}} f(z) \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} dz \qquad (w \in \mathbb{D}).$$

Note the similarity of the Berezin transform to the Poisson integral.

The question Sarason raised concerning the Bergman space is dealt with in [24,30,33,34, 36,37] and [35]. More explicitly, the question was as to when the densely defined operator $T_f T_{\overline{g}}$ is bounded on L^2_a for $f, g \in L^2_a$? While this question was originaly posed by Sarason in [31], a conjecture in section 8 of [33] more explicitly resembles Sarason's Hardy space case conjecture. This time it is conjectured that the Toeplitz product $T_f T_{\overline{g}}$ is bounded for f and $g \in L^2_a$ on the Bergman space L^2_a if and only if the product of Berezin transforms, $\widetilde{|f|^2}(w)\widetilde{|g|^2}(w)$, is uniformly bounded on $\mathbb{D}$. The question is investigated in various different cases such as the weighted Bergman space with standard weights and the Bergman space on the unit ball and polydisk. These papers prove results that approximate to the Bergman space version of Sarason's conjecture as stated in section 8 of [33]. The purpose of the first two chapters of this thesis is to investigate products of Toeplitz operators on a Bergman space of vector-valued functions, and to generalize some of the approximations to Sarason's conjecture on the Bergman space. We also give a complete characterization of Toeplitz products which are bounded and invertible on the vector-valued Bergman space, generalising the work of Stroethoff and Zheng in [34].

1.4 Toeplitz Operators on a Vector-Valued Bergman Space

For a measurable function $f : \mathbb{D} \to \mathbb{C}^n$ with $(\int_{\mathbb{D}} ||f(z)||^p_{\mathbb{C}^n} dA(z))^{\frac{1}{p}} < \infty$, we say that $f \in L^p(\mathbb{D}, \mathbb{C}^n)$. The vector-valued Bergman space, $L^p_a(\mathbb{D}, \mathbb{C}^n)$, is the intersection of $L^p(\mathbb{D}, \mathbb{C}^n)$ with the analytic $\mathbb{C}^n$- valued functions on $\mathbb{D}$ with the usual identification of functions which only differ on sets of measure 0. The norm is given by $||f||_{L^p_a(\mathbb{C}^n)} = (\int_{\mathbb{D}} ||f(z)||^p_{\mathbb{C}^n} dA(z))^{\frac{1}{p}}$, where dA is normalized Lebesgue measure on the unit disk $\mathbb{D}$. In the case $p = 2$ this space becomes a Hilbert space with the inner product given by $\langle f, g\rangle = \int_{\mathbb{D}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} dA(z)$. $L^p(\mathbb{C}^n)$ and $L^p_a(\mathbb{C}^n)$ are Banach spaces for $1 \le p < \infty$. For details see for example [2].

In the case of the vector-valued Bergman space $L^2_a(\mathbb{C}^n)$ we define the Toeplitz operator to be the densely defined composition of multiplication by an $n \times n$ matrix-valued function and the orthogonal projection from $L^2(\mathbb{C}^n)$ into $L^2_a(\mathbb{C}^n)$. So, in this case the symbol F will be a matrix of L^2 functions and $T_F v = P(Fv)$, where v is a bounded analytic $\mathbb{C}^n$-valued function. If

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots \\ f_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

and $v = (v_1, v_2, \ldots, v_n)$, where $f_{ij} \in L^2$ and $v_i \in H^\infty$, then

$$T_F v = P(Fv) = P\left(\sum_{i=1}^{n} f_{1i}v_i, \sum_{i=1}^{n} f_{2i}v_i, \ldots, \sum_{i=1}^{n} f_{ni}v_i\right) = \begin{pmatrix} T_{f_{11}} & T_{f_{12}} & \cdots \\ T_{f_{21}} & \ddots & \\ \vdots & & \end{pmatrix} v,$$

where each $T_{f_{ij}}$ is a densely defined Toeplitz operator on the scalar Bergman space L^2_a. When looking at products of these Toeplitz operators analogous to the treatment in [33] we have products of the form $T_F T_{G^*}$, where F and G are square matrices of scalar-valued Bergman space L^2_a functions.

We will now state some of the approximations to Sarason's conjecture on the Bergman space that we will generalize in the first two chapters of this thesis.

1.5 Approximations of Sarason's Conjecture on the Bergman Space

The first two theorems we generalize give a sufficient and a necessary condition for the Toeplitz product $T_f T_{\overline{g}}$ to be bounded on L^2_a. They are due to Stroethoff and Zheng and can be found in [33].

Theorem 1.5.1. *Let f and $g \in L^2_a$, then if $T_f T_{\overline{g}}$ is bounded on L^2_a,*

$$\sup_{w\in\mathbb{D}} \widetilde{|f|^2}(w)\widetilde{|g|^2}(w) < \infty.$$

Theorem 1.5.2. *Let f and $g \in L^2_a$. If*

$$\sup_{w\in\mathbb{D}} \widetilde{|f|^{2+\epsilon}}(w)\widetilde{|g|^{2+\epsilon}}(w) < \infty,$$

for some $\epsilon > 0$, then $T_f T_{\overline{g}}$ is bounded on L^2_a.

Observe that, loosely speaking, these two theorems miss the Sarason's conjecture characterization by an ϵ.

In [34] Stroethoff and Zheng went on to prove the following theorem which we also generalize:

Theorem 1.5.3. *Let f and $g \in L^2_a$. $T_f T_{\overline{g}}$ is invertible and bounded on L^2_a, if and only if*

$$\sup_{w\in\mathbb{D}} \widetilde{|f|^2}(w)\widetilde{|g|^2}(w) < \infty$$

and

$$\inf_{w\in\mathbb{D}} |f(w)||g(w)| > 0.$$

Before we continue we give some background theory.

1.6 Matrix Inequalities and Commutativity

We say that an $n \times n$ matrix A is strictly positive if, for all nonzero vectors $x \in \mathbb{C}^n$, $\langle Ax, x\rangle > 0$. Similarly, an $n\times n$ matrix A is positive if, for all nonzero vectors x, $\langle Ax, x\rangle \geq 0$. This notion of positivity allows us to define a partial order on n dimensional matrices, and also on bounded operators on a Hilbert space where the same notion of positivity also makes sense. For two matrices A and B we say that $A > B$ if $A - B$ is strictly positive, and $A \geq B$ if $A - B$ is positive. This partial ordering of matrices is sometimes referred to as Löwner partial ordering. All matrices of the form BB^* are easily seen to be positive, but it is also true that any positive matrix A can be expressed in the form BB^*. Furthermore, $||A|| = ||B||^2 = ||B^*||^2$ when $A = BB^*$. If we have a constant multiple of the identity matrix, CI where $C > 0$, then $A > CI$, for some matrix A, implies $||A|| > |C|$. Also, for positive matrices A and B, if $A < B$ then $||A|| < ||B||$. It is, however, not true in general that $||A|| < ||B||$ implies $A < B$.

A frequently occuring theme in this thesis is that something which is true for a function $f : \mathbb{R} \mapsto \mathbb{C}$ is often not true, or at least not obvious, for a function $F : \mathbb{R} \mapsto \mathbb{M}_n$, where $\mathbb{M}_n$ is the $n \times n$ complex matrices, due to the noncommutativity of matrix multiplication. For example if we have two locally integrable functions f and g, then $\int_I f(x)dx \int_I g(x)dx = \int_I g(x)dx \int_I f(x)dx$ is obviously true for $\mathbb{C}$-valued functions but false for many $\mathbb{M}_n$-valued functions. When integrating $\mathbb{M}_n$-valued functions we are just integrating each term in the matrix.

Also, either $\int_I f(x)dx \leq \int_I g(x)dx$ or $\int_I f(x)dx > \int_I g(x)dx$ is true for real-valued functions, but neither might be true for $\mathbb{M}_n$ -valued functions since matrices are not totally ordered. There are cases where we can use Löwner partial ordering to get around this problem, however, it isn't always straight forward. As we are often estimating the operator norm of matrices, the interplay between matrix inequalities and operator norm inequalities comes into play.

In some cases we can get around the problem of noncommutativity of matrices. As we are often dealing with estimates involving the operator norm of a matrix we can use the trace as an equivalent norm. While using the trace doesn't allow us to completely interchange the order of the matrices, it does allow us to carry out cyclic permutations of them.

More details on matrix inequalities may be found in [1], [40] and [41].

1.7 Matrix weighted $L^2(\mathbb{R}, \mathbb{C}^n)$

Let $L^2(\mathbb{R}, \mathbb{C}^n)$ denote the space of square-integrable $\mathbb{C}^n$-valued measurable functions,

$$\left\{ f : \mathbb{R} \to \mathbb{C}^n : \int_{\mathbb{R}} \langle f(t), f(t) \rangle_{\mathbb{C}^n} dt < \infty \right\},$$

with norm given by

$$||f||_{L^2(\mathbb{R},\mathbb{C}^n)} = \left(\int_{\mathbb{R}} \langle f(t), f(t) \rangle_{\mathbb{C}^n} dt \right)^{\frac{1}{2}}.$$

For a matrix valued function V which is positive, measurable, invertible almost everywhere and locally integrable, let $L^2(\mathbb{R}, \mathbb{C}^n, V)$ be the space of measurable functions,

$$\left\{ f : \mathbb{R} \to \mathbb{C}^n : \int_{\mathbb{R}} \langle V(t)^{\frac{1}{2}} f(t), V(t)^{\frac{1}{2}} f(t) \rangle dt < \infty \right\},$$

with norm

$$||f||_{L^2(\mathbb{R},\mathbb{C}^n,V)} = \left(\int_{\mathbb{R}} \langle V(t)^{\frac{1}{2}} f(t), V(t)^{\frac{1}{2}} f(t) \rangle dt \right)^{\frac{1}{2}}.$$

This generalizes the notion of weighted L^2 spaces of scalar functions, where a weight is a measurable, locally integrable, almost everywhere positive function. We will generally denote $L^2(\mathbb{R}, \mathbb{C}^n, V)$ by $L^2(V)$.

Definition 1.7.1. We refer to matrix valued functions which are measurable, locally integrable, almost everywhere positive and invertible as *matrix weights*.

1.8 The Hilbert Transform and Singular Integral Operators

One of the most studied, and most important, operators in harmonic analysis is the Hilbert transform. The Hilbert transform, H, is the singular integral operator of convolution type with kernel $\frac{1}{x}$: for $f \in L^2(\mathbb{R})$,

$$Hf(y) = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{f(y-x)}{x} dx.$$

Primarily studied on the function space $L^p(\mathbb{R}, \mathbb{C})$, an obvious question to ask is as to when the Hilbert transform is bounded on the function space $L^p(\mathbb{R}, \mathbb{C}, \mu)$ for some measure μ. One can prove that the measure has to be absolutely continuous, i.e. of the form $\mu = w(x)dx$ for some locally integrable positive function w, see [10]. The problem of characterizing positive locally integrable functions w such that the Hilbert transform is bounded on $L^2(\mathbb{R}, \mathbb{C}, w)$ we initially solved by Helson and Szegö in [13]. A different characterization, as well as a generalization for $L^p(\mathbb{R}, \mathbb{C}, w)$, was then given by Hunt, Muckenhoupt and Wheeden. This characterization is in terms of the A_p condition:

$$\sup_I \langle w \rangle_I \left\langle \frac{1}{w^{\frac{1}{p-1}}} \right\rangle_I^{p-1} < \infty, \tag{1.8.1}$$

where the supremum is over intervals I. Thus we have the following famous theorem due to Hunt, Muckenhoupt and Wheeden [14].

Theorem 1.8.1. *For $1 < p < \infty$ and a weight w, there exists $C > 0$ with*

$$\int |(Hf)(x)|^p w(x)dx \leq C \int |f(x)|^p w(x)dx$$

if and only if w satisfies the A_p condition, (1.8.1). The constant C is dependent only on p and the A_p constant of w.

The Hilbert transform often serves as the canonical example of a more general class of operators called singular integral operators. The Hunt-Muckenhoupt-Wheeden condition also characterizes weights for which Calderón-Zygmund singular integral operators are bounded $L^p(w) \to L^p(w)$.

A natural question to ask, given a characterization of measures such that $H : L^p(\mathbb{R}, \mathbb{C}, w) \mapsto L^p(\mathbb{R}, \mathbb{C}, w)$ is bounded, is whether there is a characterization of two locally integrable functions v and u such that $H : L^p(\mathbb{R}, \mathbb{C}, v) \mapsto L^p(\mathbb{R}, \mathbb{C}, u)$ is a bounded operator. This is a famous open problem in harmonic analysis and much progress has been made towards solving in recently. This includes characterizations of weights such that various dyadic models of singular integral operators as well as truncated singular integral operators are bounded $L^p(\mathbb{R}, \mathbb{C}, v) \mapsto L^p(\mathbb{R}, \mathbb{C}, u)$.

An immediate conjecture was that a two weight Hunt-Muckenhoupt-Wheeden condition would characterize such weights but this was shown to be false for the Hilbert transform.

$$\text{Two-weight } A_p \text{ condition:} \qquad \sup_I \langle u \rangle_I \left\langle \frac{1}{v^{\frac{1}{p-1}}} \right\rangle_I^{p-1} < \infty.$$

On vector-valued function spaces the Hilbert transform has also been studied, and an analogous condition to the original Hunt-Muckenhoupt-Wheeden condition was found by Sergei Treil and Alexander Volberg in [38] for $p = 2$. This result was extended to $1 < p < \infty$ in [21] by Fedja Nazarov and Sergei Treil. The theory has also been generalized for Calderon–Zygmund singular integral operators in [21].

In contrast, little attention has been given so far to understanding two-weight problems on vector-valued function spaces (the work of C. M. Pereyra and N. H. Katz [26] being the only exception.) The purpose of chapter 4 is to find sufficient conditions for the Hilbert transform to be bounded $L^2(\mathbb{R}, \mathbb{C}^n, V) \mapsto L^2(\mathbb{R}, \mathbb{C}^n, U)$ by first looking at certain dyadic operators including the martingale transform and the dyadic shift.

We also look at some singular integral operators of convolution type satisfying certain smoothness estimates on the kernel and show that these are also bounded given the same hypothesis.

1.9 Martingale Operators, the Dyadic Shift, Band Operators Modelling the Hilbert Transform and some Singular Integral Operators

Let $\mathcal{D}$ denote the standard grid of dyadic subintervals of $\mathbb{R}$, where n and k range over the integers $\mathcal{D} = \{[k2^{-n}, (k+1)2^{-n})\}$. The Haar functions associated to a dyadic interval I are defined as $h_I = \frac{1}{\sqrt{|I|}}\left(\chi_{I_-} - \chi_{I_+}\right)$, where $I_-$and I_+ are the largest proper dyadic subintervals of I on the right and the left respectively. The $\{h_I\}_{I\in\mathcal{D}}$ form an orthonormal basis for $L^2(\mathbb{R})$. If we take an orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$, then the $L^2(\mathbb{R}, \mathbb{C}^n)$ vectors $\{h_I e_i\}_{I\in\mathcal{D}, i=1\ldots n}$ form an orthonormal basis of $L^2(\mathbb{R}, \mathbb{C}^n)$.

We firstly consider the operator T_σ on $L^2(\mathbb{R}, \mathbb{C}^n)$ defined by the mapping

$$T_\sigma f \mapsto \sum_{I\in\mathcal{D}} \sigma(I) h_I f_I,$$

where $f_I = \int_I f h_I$ and $\sigma(I) = \pm 1$. The T_σ are *dyadic martingale transforms* and are unitary operators on $L^2(\mathbb{R}, \mathbb{C}^n)$. Martingale transforms often serve as models for singular integral operators.

A sufficient condition for uniform boundedness of the T_σ in the operator one weight case has been given in [29], and in [11] it is shown that the dyadic operator A_2 condition (infinite dimensional) does not imply the boundedness of the martingale transforms. We will, however, be concerned with the two weight matrix case.

Chapter 4 of this thesis will mainly discuss conditions on a pair of matrix weights, U and V, which imply that the martingale transforms are uniformly bounded from $L^2(\mathbb{R}, \mathbb{C}^n, V)$ to $L^2(\mathbb{R}, \mathbb{C}^n, U)$. This is equivalent to showing that the operators $M_U^{\frac{1}{2}} T_\sigma M_V^{-\frac{1}{2}}$ are uniformly bounded on the unweighted space $L^2(\mathbb{R}, \mathbb{C}^n)$. The sufficient conditions we find on a pair of matrix weights are a joint A_2 condition, a matrix A_∞ condition on one weight and a matrix reverse Hölder condition on the other weight. We can also, as a corollary, replace the matrix reverse Hölder condition by the matrix A_∞ condition. The initial intention was to use this operator as a model for the Hilbert transform and other operators. It turns out we have to consider another type of operator best understood by its action on the Haar basis, the dyadic shift. From this, we will deduce the boundedness of the Hilbert transform by way of Petermichl's explicit formulation of the Hilbert transform in terms of averages of Haar shifts. We then take this a step further and show that under our hypothesis we also obtain boundedness of a more general class of operators defined in terms of how they act on the Haar basis, the so called band operators. This time we employ a generalization of Petermichl's work due to Vagharshakyan that models more general singular integral operators in terms of certain band operators.

Definition 1.9.1. The *dyadic shift* Ш with respect to the standard dyadic grid is the operator given by

$$Ш f = Ш \sum_{I \in \mathcal{D}} f_I h_I = \sum_{I \in \mathcal{D}} \langle f, h_{I_+} - h_{I_-} \rangle h_I,$$

where f is supported on the unit interval and has finite Haar expansion.

What Petermichl showed was that if we average dyadic shift operators defined on different scalings and translations of the dyadic intervals then we recover the Hilbert transform. We will use a refinement of this result due to Hytönen [15].

Definition 1.9.2. Let $\beta = \{\beta_n\}$ be a sequence, indexed by $\mathbb{Z}$, of elements from the set $\{0, 1\}$, and let $1 \leq r < 2$. Then the dyadic grid $\mathbb{D}_{r,\beta}$ is defined to be the collection of scaled and translated dyadic intervals

$$\mathbb{D}_{r,\beta} = \left\{ r 2^m \left([0,1) + l + \sum_{i<m} 2^{i-m} \beta_i \right) \right\}_{l,m \in \mathbb{Z}}.$$

This definition has been taken explicitly from Vagharshakyan's preprint [39]. It is also constructed in [15].

The dyadic shift with respect to this grid will be the operator

$$Ш^{\beta,r} f = Ш^{\beta,r} \sum_{I \in \mathbb{D}^{\beta,r}} f_I h_I = \sum_{I \in \mathbb{D}^{\beta,r}} \langle f, h_{I_+} - h_{I_-} \rangle h_I, \tag{1.9.1}$$

where f is supported on the interval $(r\beta, r(1+\beta))$ and has finite Haar expansion.

1.9.1 The Theorem of Petermichl

Theorem 1.9.3. *[Petermichl, reformulation by Hytönen]*

If $\mathbb{P}$ is the probability measure on the space of sequences $\{0,1\}^{\mathbb{Z}}$, of elements from $\{0, 1\}$, such that the probability that $\beta_l = 0$ is $\frac{1}{2}$ and the probability that $\beta_l = 1$ is also $\frac{1}{2}$ for each l, then the Hilbert transform is a positive constant multiple of the following operator:

$$f \to \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 Ш^{\beta,r} f \frac{dr}{r} d\mathbb{P}(\beta).$$

Proof. See [15] and [39]. □

1.10 The A_p and A_∞ Conditions on Scalar Weights

One of the key concepts throughout this thesis will be matrix generalizations of A_p weights and A_∞ weights. As we have seen, a weight satisfies the A_p condition or is in the A_p class if

$$\sup_{I=(a,b),(a,b)\neq\varnothing} \langle w \rangle_I \left\langle \frac{1}{w^{\frac{1}{p-1}}} \right\rangle_I^{p-1} \leq C_p < \infty.$$

We will mainly be concerned with generalizations of the A_2 condition, which notationally has a nicer aesthetic,

$$\sup_{I=(a,b),(a,b)\neq\varnothing} \langle w \rangle_I \left\langle w^{-1} \right\rangle_I \leq C_2 < \infty.$$

A crucial fact is that if $w \in A_p$ then $w \in A_q$ for all $q > p$ and $C_q \leq C_p$, see for example [32] page 195. Another class of weight functions we will be concerned with generalizing is A_∞:

Definition 1.10.1. $w \in A_\infty$ if there exists a constant C such that for all intervals I

$$\langle w \rangle_I \leq C \exp\langle \log w \rangle_I.$$

A useful fact about A_∞ weights is contained in the following theorem, the proof of which can be found in [32] on page 212.

Theorem 1.10.2. *A_∞ is the union of all A_p classes:*

$$A_\infty = \bigcup_{p<\infty} A_p.$$

1.11 The Two-Weight Problem and Toeplitz Products

We reproduce the following commutative diagram from [5] that illuminates the connection between the boundedness of Toeplitz products on the Hardy space and the two weight problem.

$$\begin{tikzcd} H^2 \arrow[r, "T_f T_{\overline{g}}"] \arrow[d, "M_{\overline{g}}"] & H^2 \\ L^2(\frac{1}{|g|^2}) \arrow[r, "P"] & L^2(|f|^2) \arrow[u, "M_f"'] \end{tikzcd}$$

Here P is the projection from L^2 onto H^2, or rather the integral operator that defines this, which will be well defined on $L^2(\frac{1}{|g|^2})$. Note that here M_f and $M_{\overline{g}}$ are isometries and so the boundedness of P will imply the boundedness of $T_f T_{\overline{g}}$. The direct connection with the two-weight problem then follows from the fact that the boundedness of the projection P and the Hilbert transform H imply each other here.

Chapter 2

Vector Toeplitz Products

2.0.1 Main Theorems

The first two main theorems follow, one giving a sufficient condition for the Toeplitz product $T_F T_{G^*}$ to be bounded and the other a necessary condition. Both are conditions involving the Berezin transform:

Definition 2.0.1. The Berezin transform of a matrix-valued L^2 function is the matrix-valued function $B(A)$, where $B(A)(w) = \int (A \circ \phi_w)(z) dA(z)$ for $w \in \mathbb{D}$. Composition here being composition with each matrix entry. Here, ϕ_w is the Möbius transform $z \mapsto \frac{w-z}{1-\overline{w}z}$. We should also note here that $B(A)(w) = \int A(z) \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} dA(z)$ by a change of variables. Defining the normalized reproducing kernel $k_w(z)$ as $\frac{K_w(z)}{||K_w||}$, we obtain $|k_w(z)|^2 = \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}$.

Here is our first main result:

Theorem 2.0.2. *Let F and G be matrices of L^2_a functions and let $\epsilon > 0$.*
If $\operatorname{tr}\left(B((F^*F)^{\frac{2+\epsilon}{2}})(w)B((G^*G)^{\frac{2+\epsilon}{2}})(w)\right)$ *is uniformly bounded over all $w \in \mathbb{D}$, then the Toeplitz product $T_F T_{G^*}$ is bounded $L^2_a(\mathbb{C}^n) \to L^2_a(\mathbb{C}^n)$.*

We also have the following condition: If there exists $\epsilon > 0$ and $C > 0$ such that

$$\sup_{w \in \mathbb{D}} \left(\int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} (\operatorname{tr}(G(z)F(x)^*F(x)G(z)^*))^{\frac{2+\epsilon}{2}} |k_w(z)|^2 dA(z) \right\} |k_w(x)|^2 dA(x) \right)^{\frac{1}{2+\epsilon}} < C,$$

then the Toeplitz product $T_F T_{G^*} : L^2_a(\mathbb{C}^n) \to L^2_a(\mathbb{C}^n)$ is bounded.

Here is the necessary condition:

Theorem 2.0.3. *If the product of Toeplitz operators $T_F T_{G^*}$ is bounded on $L^2_a(\mathbb{C}^n)$, then* $\sup_{w \in \mathbb{D}} \operatorname{tr}(B(F^*F)(w)B(G^*G)(w)) < \infty$.

2.1 Bounded Toeplitz Products

2.1.1 A Sufficient Condition(Proof of Theorem 2.0.2)

The technique in [33] for showing a sufficient condition on the boundedness of a Toeplitz product involves an inner product formula that easily generalizes to the vector-valued case. So for $g, f \in L^2_a(\mathbb{C}^n)$

$$\langle f, g\rangle_{L^2_a(\mathbb{C}^n)} = \int_{\mathbb{D}} \langle f(z), g(z)\rangle_{\mathbb{C}^n}\, dA(z) = 3\int_{\mathbb{D}} (1-|z|^2)^2 \langle f(z), g(z)\rangle_{\mathbb{C}^n}\, dA(z) +$$

$$\frac{1}{2}\int_{\mathbb{D}} (1-|z|^2)^2 \left\langle f'(z), g'(z)\right\rangle_{\mathbb{C}^n} dA(z) + \frac{1}{3}\int_{\mathbb{D}} (1-|z|^2)^3 \left\langle f'(z), g'(z)\right\rangle_{\mathbb{C}^n} dA(z).$$

So, to estimate the norm of $T_G T_F^*$ we will look at the inner product $\langle T_G T_F^* u, v\rangle_{L^2_a(\mathbb{C}^n)}$, with $v, u \in L^2_a(\mathbb{C}^n)$, in the form just given.

Let us start by estimating the term $\langle T_{F^*}(u)(w), T_{G^*}(v)(w)\rangle_{\mathbb{C}^n}$ for $w \in \mathbb{D}$.

Definition 2.1.1. For $f, g \in L^2(\mathbb{D})$, define the rank 1 operator $f \otimes g : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ by

$$(f \otimes g)h = \langle h, g\rangle f$$

for $h \in L^2(\mathbb{D})$.

Also for $F, G \in M_{n\times n}(L^2(\mathbb{D}))$, define the operator $F \otimes G : L^2(\mathbb{D}, \mathbb{C}^n) \to L^2(\mathbb{D}, \mathbb{C}^n)$ by

$$(F \otimes G)h = \begin{pmatrix} \sum_{i=1}^n f_{1i} \otimes g_{1i} & \sum_{i=1}^n f_{1i} \otimes g_{2i} & \cdots & \sum_{i=1}^n f_{1i} \otimes g_{ni} \\ \sum_{i=1}^n f_{2i} \otimes g_{1i} & \cdots & & \\ \vdots & \ddots & & \\ & & & \\ \sum_{i=1}^n f_{ni} \otimes g_{1i} & \sum_{i=1}^n f_{ni} \otimes g_{2i} & \cdots & \sum_{i=1}^n f_{ni} \otimes g_{ni} \end{pmatrix} h$$

for $h \in L^2(\mathbb{D}, \mathbb{C}^n)$.

Theorem 2.1.2. *For F,$G \in M_{n\times n}(L^2_a(\mathbb{D})$ and $w \in \mathbb{D}$,*

$$\begin{aligned} &\langle T_{F^*}(u)(w), T_{G^*}(v)(w)\rangle_{\mathbb{C}^n} \\ &= \frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}} \langle (Gk_w \otimes Fk_w)u(z), v(z)\rangle_{\mathbb{C}^n}\, dA(z), \end{aligned}$$

where k_w is the normalized reproducing kernel.

Proof.

$$\begin{aligned}
&\langle T_{F^*}(u)(w), T_{G^*}(v)(w)\rangle_{\mathbb{C}^n}\\
&= \left\langle \int_{\mathbb{D}} F^*(z)u(z)\overline{K_w(z)}dA(z), \int_{\mathbb{D}} G^*(\zeta)v(\zeta)\overline{K_w(\zeta)}dA(\zeta)\right\rangle_{\mathbb{C}^n}
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{D}}\int_{\mathbb{D}} \langle G(\zeta)K_w(\zeta)\left(F(z)K_w(z)\right)^* u(z), v(\zeta)\rangle_{\mathbb{C}^n} dA(z)dA(\zeta)\\
&= \frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \langle G(\zeta)k_w(\zeta)\left(F(z)k_w(z)\right)^* u(z), v(\zeta)\rangle_{\mathbb{C}^n} dA(z)dA(\zeta)\\
&= \frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}} \langle (Gk_w \otimes Fk_w u)(\zeta), v(\zeta)\rangle_{\mathbb{C}^n} dA(\zeta).
\end{aligned}$$

□

Lemma 2.1.3.

$$\begin{aligned}
&||(F\otimes G)(F\otimes G)^*||_{op} \le \operatorname{tr}\{(F\otimes G)(G\otimes F)\}\\
&= \sum_{q=1}^{n}\sum_{m=1}^{n}\sum_{r=1}^{n}\sum_{l=1}^{n} \langle f_{qr}, f_{ql}\rangle_{L^2}\langle g_{ml}, g_{mr}\rangle_{L^2}.
\end{aligned}$$

Proof. Firstly note that $(F\otimes G)^* = G\otimes F$. As $(F\otimes G)(G\otimes F)$ is of finite rank, the trace of $(F\otimes G)(G\otimes F)$ will be an equivalent norm. We can express $F\otimes G$ as a matrix of operators on the scalar Bergman space with the entries $[\sum_{l=1}^n f_{il}\otimes g_{jl}]_{i,j}$. We can then express $(F\otimes G)(G\otimes F)$ in a similar manner:

$$\begin{aligned}
&\left[\sum_{m=1}^{n}\left(\sum_{l=1}^{n} f_{il}\otimes g_{ml}\right)\left(\sum_{l=1}^{n} g_{ml}\otimes f_{jl}\right)\right]_{i,j}\\
&= \left[\sum_{m=1}^{n}\left(\sum_{r=1}^{n}\sum_{l=1}^{n}\langle\cdot, f_{jl}\rangle_{L^2}\langle g_{ml}, g_{mr}\rangle_{L^2} f_{ir}\right)\right]_{i,j}.
\end{aligned}$$

Note that we have as an orthonormal basis $e_{l,m} = (0,\ldots,0,z^l\sqrt{l+1},0,\ldots)$, i.e. a vector with each coordinate 0 apart from the mth entry, which is the lth orthonormal basis element of the scalar-valued Bergman space. So the trace of the operator $(F\otimes G)(G\otimes F)$ will be

$$\begin{aligned}
&\sum_{p,q}\langle (F\otimes G)(G\otimes F)e_{p,q}, e_{p,q}\rangle\\
&= \sum_{q=1}^{n}\sum_{p=0}^{\infty}\sum_{m,r,l}\left\langle z^p\sqrt{1+p}, f_{ql}\right\rangle_{L^2}\langle g_{ml}, g_{mr}\rangle_{L^2}\int_{\mathbb{D}} f_{qr}(z)\overline{z^p\sqrt{1+p}}dA(z).
\end{aligned}$$

We can write each f_{ij} as a power series $\sum_{s=0}^{\infty} a_{s,ij} z^s \sqrt{1+s}$, thus this trace becomes

$$\sum_{q=1}^{n}\sum_{p=0}^{\infty}\sum_{m=1}^{n}\left(\sum_{r=1}^{n}\sum_{l=1}^{n}\overline{a_{p,ql}}\left\langle g_{ml}, g_{mr}\right\rangle_{L^2} a_{p,qr}\right),$$

and thus by Parseval's identity

$$\sum_{q=1}^{n}\sum_{m=1}^{n}\left(\sum_{r=1}^{n}\sum_{l=1}^{n}\left\langle f_{qr}, f_{ql}\right\rangle_{L^2}\left\langle g_{ml}, g_{mr}\right\rangle_{L^2}\right).$$

□

Theorem 2.1.4. *There exist dimension dependent constants c and C such that*

$$c(\mathrm{tr}(B(G^*G)(w)B(F^*F)(w)))^{\frac{1}{2}} \leq ||Gk_w \otimes Fk_w||_{op} \leq C(\mathrm{tr}(B(G^*G)(w)B(F^*F)(w)))^{\frac{1}{2}}.$$

Proof. We can see that $||Gk_w \otimes Fk_w||_{op}$ is equivalent to the square root of the trace of the operator $(Gk_w \otimes Fk_w)(Gk_w \otimes Fk_w)^*$, and hence Lemma 2.1.3 implies that this is equal to

$$\begin{aligned}&\sum_{q=1}^{n}\sum_{m=1}^{n}\left(\sum_{r=1}^{n}\sum_{l=1}^{n}\left\langle f_{qr}, f_{ql}|k_w|^2\right\rangle_{L^2}\left\langle g_{ml}, g_{mr}|k_w|^2\right\rangle_{L^2}\right)\\&= \mathrm{tr}(B(G^*G)(w)B(F^*F)(w)).\end{aligned}$$

□

Definition 2.1.5. The operator P_0 defined on $L_p(\mathbb{D})$ is the operator that sends $f \in L^2$ to the function given by $(P_0 f)(w) = \int_{\mathbb{D}} \frac{f(z)}{|1-\overline{w}z|^2} dA(z)$.

Elements from the following two theorems are borrowed from Lemma 3.2 in [33].

Lemma 2.1.6. *If we have a scalar-valued integrable function h and a scalar-valued Bergman space function v, then, for each $w \in \mathbb{D}$ and $\epsilon > 0$,*

$$\begin{aligned}&\int_{\mathbb{D}}\left|\frac{\overline{h(x)}|v(x)|}{(1-\overline{x}w)^3}\right| dA(x)\\&\leq 2\frac{1}{1-|w|^2}\left\{\int_{\mathbb{D}}|h(x)|^{2+\epsilon}|k_w(x)|^2 dA(x)\right\}^{\frac{1}{2+\epsilon}}\left\{(P_0|v|^{\delta})(w)\right\}^{\frac{1}{\delta}},\end{aligned}$$

where $\delta = \frac{2+\epsilon}{1+\epsilon}$.

Proof. By Hölder's inequality,

$$\int_{\mathbb{D}}\left|\frac{\overline{h(x)}|v(x)|}{(1-\overline{x}w)^3}\right|dA(x) \leq \int_{\mathbb{D}}\frac{|h(x)||1-\overline{x}w||v(x)|}{|1-\overline{x}w|^4}dA(x)$$

$$\leq \left\{\int_{\mathbb{D}}\frac{|h(x)|^{2+\epsilon}}{|1-\overline{x}w|^4}dA(x)\right\}^{\frac{1}{2+\epsilon}}\left\{\int_{\mathbb{D}}\frac{|1-\overline{x}w|^{\delta}|v(x)|^{\delta}}{|1-\overline{x}w|^4}dA(x)\right\}^{\frac{1}{\delta}}$$

$$= \frac{1}{1-|w|^2}\left\{\int_{\mathbb{D}}|h(x)|^{2+\epsilon}\frac{(1-|w|^2)^2}{|1-\overline{x}w|^4}dA(x)\right\}^{\frac{1}{2+\epsilon}}\left\{\int_{\mathbb{D}}\frac{|1-|w|^2|^{\frac{\epsilon}{1+\epsilon}}|v(x)|^{\delta}}{|1-\overline{x}w|^2|1-\overline{x}w|^{\frac{\epsilon}{\epsilon+1}}}dA(x)\right\}^{\frac{1}{\delta}}$$

$$= \frac{1}{1-|w|^2}\left\{\int_{\mathbb{D}}|h(x)|^{2+\epsilon}|k_w(x)|^2dA(x)\right\}^{\frac{1}{2+\epsilon}}\left\{\int_{\mathbb{D}}\frac{|1-|w|^2|^{\frac{\epsilon}{1+\epsilon}}|v(x)|^{\delta}}{|1-\overline{x}w|^2|1-\overline{x}w|^{\frac{\epsilon}{\epsilon+1}}}dA(x)\right\}^{\frac{1}{\delta}},$$

and our result follows from the fact that

$$\begin{aligned}\frac{1-|w|^2}{|1-\overline{w}z|} &\leq \frac{(1-|w|)(1+|w|)}{|1-\overline{w}z|} \leq \frac{(1-|w|)(1+|w|)}{|1-|w||z||}\\ &\leq \frac{(1-|w|)(1+|w|)}{|1-|w||} \leq 1+|w| < 2.\end{aligned}$$

□

Let us now take a look at $\langle T_{F^*}(u)'(w), T_{G^*}(v)'(w)\rangle_{\mathbb{C}^n}$.

Theorem 2.1.7. *Let $\epsilon > 0$ and $\frac{1}{\delta} = 1 - \frac{1}{2+\epsilon}$, then there exist constants $C_1, C_2 > 0$ such that*

$$\begin{aligned}&\left|\langle T_{F^*}(u)'(w), T_{G^*}(v)'(w)\rangle_{\mathbb{C}^n}\right|\\ &\leq C_1\left(\int_{\mathbb{D}}\int_{\mathbb{D}}(\operatorname{tr}(G(z)F(x)^*F(x)G(z)^*))^{\frac{2+\epsilon}{2}}|k_w(z)|^2dA(z)|k_w(x)|^2dA(x)\right)^{\frac{1}{2+\epsilon}}\frac{1}{(1-|w|^2)^2}\\ &\qquad\times\left\{(P_0||u||^{\delta}_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||v||^{\delta}_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\\ &\leq C_2\left(\left\{\operatorname{tr}(B((F^*F)^{\frac{2+\epsilon}{2}})(w)B((G^*G)^{\frac{2+\epsilon}{2}})(w))\right\}^{\frac{1}{2+\epsilon}}\right)\frac{1}{(1-|w|^2)^2}\\ &\qquad\times\left\{(P_0||u||^{\delta}_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||v||^{\delta}_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\end{aligned}$$

for all $w \in \mathbb{D}$.

Proof. First note that for each function $u \in L^2_a(\mathbb{D}, \mathbb{C}^n)$ and $w \in \mathbb{D}$

$$\langle u, K'_w\rangle = u'(w),$$

and so

$$
\begin{aligned}
&|\left\langle T_{F^*}(u)'(w), T_{G^*}(v)'(w)\right\rangle_{\mathbb{C}^n}| \\
&= \left| \int\int \left\langle F^*(z)u(z)\overline{K'_w(z)}, G^*(x)v(x)\overline{K'_w(x)}\right\rangle_{\mathbb{C}^n} dA(z)dA(x)\right| \\
&\leq \left| \int\int ||G(z)F^*(x)||_{M_{n\times n}}||u(z)||||v(x)|| \left|\frac{K_w(z)}{1-\overline{w}z}\right| \left|\frac{K_w(x)}{1-\overline{w}x}\right| dA(z)dA(x)\right|.
\end{aligned}
$$

If we denote the function v in Lemma 2.1.6 by g, then $g(z) = ||u(z)||$, and with $h(z) = ||G(z)F^*(x)||_{M_{n\times n}}||v(x)|| \left|\frac{K_w(x)}{1-\overline{w}x}\right|$, applying Lemma 2.1.6 we arrive at the following inequality:

$$
\begin{aligned}
&|\left\langle T_{F^*}(u)'(w), T_{G^*}(v)'(w)\right\rangle_{\mathbb{C}^n}| \\
&\leq 2\frac{1}{1-|w|^2}\left| \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \left(||G(z)F^*(x)||_{M_{n\times n}}||v(x)|| \left|\frac{K_w(x)}{1-\overline{w}x}\right|\right)^{2+\epsilon} |k_w(z)|^2 dA(z)\right\}^{\frac{1}{2+\epsilon}} \right. \\
&\qquad\qquad \left. \times \left\{(P_0||u||^{\delta})(w)\right\}^{\frac{1}{\delta}} dA(x)\right| \\
&= 2\frac{1}{1-|w|^2}\left| \int_{\mathbb{D}} \left\{ \int_{\mathbb{D}} \left(||G(z)F^*(x)||_{M_{n\times n}}\right)^{2+\epsilon} |k_w(z)|^2 dA(z)\right\}^{\frac{1}{2+\epsilon}} \right. \\
&\qquad\qquad \left. \times ||v(x)|| \left|\frac{K_w(x)}{1-\overline{w}x}\right| dA(x)\right| \\
&\qquad\qquad \times \left\{(P_0||u||^{\delta})(w)\right\}^{\frac{1}{\delta}},
\end{aligned}
$$

where $\epsilon > 0$ and $\frac{1}{\delta} = 1 - \frac{1}{2+\epsilon}$.

Again using Lemma 2.1.6, but this time with

$$
h(x) = \left\{ \int_{\mathbb{D}} \left(||G(z)F^*(x)||_{\mathbb{C}^n}\right)^{2+\epsilon} |k_w(z)|^2 dA(z)\right\}^{\frac{1}{2+\epsilon}}
$$

and $g(x) = ||v(x)||$, we see that

$$
\begin{aligned}
&\left\langle T_{F^*}(u)'(w), T_{G^*}(v)'(w)\right\rangle_{\mathbb{C}^n} \\
&\leq 4\frac{1}{(1-|w|^2)^2}\Bigg|\left\{\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}\left(||G(z)F^*(x)||_{M_{n\times n}}\right)^{2+\epsilon}|k_w(z)|^2 dA(z)\right\}|k_w(x)|^2 dA(x)\right\}^{\frac{1}{2+\epsilon}} \\
&\qquad\qquad\times\left\{(P_0||v||^\delta)(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||u||^\delta)(w)\right\}^{\frac{1}{\delta}}\Bigg| \\
&\leq 4\frac{1}{(1-|w|^2)^2}\left\{\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}(C\operatorname{tr}(G(x)F(z)^*F(z)G(x)^*))^{\frac{2+\epsilon}{2}}|k_w(z)|^2 dA(z)\right\}\right. \\
&\qquad\qquad\left.\times|k_w(x)|^2 dA(x)\right\}^{\frac{1}{2+\epsilon}} \\
&\qquad\qquad\times\left\{(P_0||u||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||v||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}.
\end{aligned}
$$

This is what we want, but we can go a step further and get something that superficially looks more similar to the analogous result in the scalar case.

Letting the 4 be absorbed into the constant C and using an inequality on matrix norms from [1], Theorem IX.2.10 on page 258, we can estimate the above expression by

$$
\begin{aligned}
&\frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}(C||(G(x)F(z)^*F(z)G(x)^*)||_{\mathbb{C}^n op})^{\frac{2+\epsilon}{2}}|k_w(z)|^2 dA(z)\right\} \\
&\qquad\times|k_w(x)|^2 dA(x)^{\frac{1}{2+\epsilon}}\left\{(P_0||u||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||v||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}} \\
&\leq\frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}(C||((G^*G(x))^{\frac{1}{2}}F^*F(z)(G^*G(x))^{\frac{1}{2}})||_{\mathbb{C}^n op})^{\frac{2+\epsilon}{2}}|k_w(z)|^2 dA(z)\right\} \\
&\qquad\times|k_w(x)|^2 dA(x)^{\frac{1}{2+\epsilon}}\left\{(P_0||u||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||v||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}} \\
&\leq\frac{1}{(1-|w|^2)^2}\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}C||(G^*G(x))^{\frac{2+\epsilon}{4}}(F^*F(z))^{\frac{2+\epsilon}{2}}(G^*G(x))^{\frac{2+\epsilon}{4}}||_{\mathbb{C}^n op}|k_w(z)|^2 dA(z)\right\} \\
&\qquad\times|k_w(x)|^2 dA(x)^{\frac{1}{2+\epsilon}}\left\{(P_0||u||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\left\{(P_0||v||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}} \\
&\leq\left(C\operatorname{tr}\left(B\left((F^*F)^{\frac{2+\epsilon}{2}}\right)(w)B\left((G^*G)^{\frac{2+\epsilon}{2}}\right)(w)\right)^{\frac{1}{2+\epsilon}}\right)\frac{1}{(1-|w|^2)^2} \\
&\qquad\times\left\{(P_0||u||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}\times\left\{(P_0||v||^\delta_{\mathbb{C}^n})(w)\right\}^{\frac{1}{\delta}}
\end{aligned}
$$

where B is the Berezin transform and C is a constant that is possibly different from line to line. □

Now let us use the estimates from Theorems 2.1.2 and 2.1.7 in the inner product formula. Taking our inner product formula

$$
\begin{aligned}
\langle T_{F^*}(u), T_{G^*}(v)\rangle_{L^2_a(\mathbb{C}^n)} &= \int_{\mathbb{D}} \langle T_{F^*}(u), T_{G^*}(v)\rangle_{\mathbb{C}^n} \, dA(z) \\
&= 3\int_{\mathbb{D}} (1-|z|^2)^2 \langle T_{F^*}(u), T_{G^*}(v)\rangle_{\mathbb{C}^n} \, dA(z) \\
&+ \frac{1}{2}\int_{\mathbb{D}} (1-|z|^2)^2 \left\langle T'_{F^*}(u), T'_{G^*}(v)\right\rangle_{\mathbb{C}^n} dA(z) \\
&+ \frac{1}{3}\int_{\mathbb{D}} (1-|z|^2)^3 \left\langle T'_{F^*}(u), T'_{G^*}(v)\right\rangle_{\mathbb{C}^n} dA(z),
\end{aligned}
$$

let's take the term $\frac{1}{2}\int_{\mathbb{D}}(1-|z|^2)^2 \langle T'_{F^*}(u), T'_{G^*}(v)\rangle_{\mathbb{C}^n} \, dA(z)$ and estimate its modulus:

$$
\begin{aligned}
&\left|\frac{1}{2}\int_{\mathbb{D}} (1-|z|^2)^2 \left\langle T'_{F^*}(u), T'_{G^*}(v)\right\rangle_{\mathbb{C}^n} dA(z)\right| \\
&\leq \frac{1}{2}\int_{\mathbb{D}} \left(C \operatorname{tr}\left(B\left((F^*F)^{\frac{2+\epsilon}{2}}\right)(w) B\left((G^*G)^{\frac{2+\epsilon}{2}}\right)(w)\right)\right)^{\frac{1}{2+\epsilon}} \frac{(1-|w|^2)^2}{(1-|w|^2)^2} \\
&\qquad\qquad \times \left\{P_0||u||^{\delta}_{\mathbb{C}^n}(w)\right\}^{\frac{1}{\delta}} \left\{P_0||v||^{\delta}_{\mathbb{C}^n}(w)\right\}^{\frac{1}{\delta}} dA(w)| \\
&\leq \frac{1}{2}\sup_{w\in\mathbb{D}} \left(C \operatorname{tr}\left(B\left((F^*F)^{\frac{2+\epsilon}{2}}\right)(w) B\left((G^*G)^{\frac{2+\epsilon}{2}}\right)(w)\right)\right)^{\frac{1}{2+\epsilon}} \\
&\qquad \times \int_{\mathbb{D}} \left\{P_0||u||^{\delta}_{\mathbb{C}^n}(w)\right\}^{\frac{1}{\delta}} \left\{P_0||v||^{\delta}_{\mathbb{C}^n}(w)\right\}^{\frac{1}{\delta}} dA(w).
\end{aligned}
$$

By Cauchy-Schwarz, this expression will be less than or equal to

$$
\begin{aligned}
&\frac{1}{2}\sup_{w\in\mathbb{D}} \left(C \operatorname{tr}\left(B\left((F^*F)^{\frac{2+\epsilon}{2}}\right)(w) B\left((G^*G)^{\frac{2+\epsilon}{2}}\right)(w)\right)\right)^{\frac{1}{2+\epsilon}} \\
&\times \left\{\int_{\mathbb{D}} \left\{P_0||u||^{\delta}_{\mathbb{C}^n}(w)\right\}^{\frac{2}{\delta}} dA(w)\right\}^{\frac{1}{2}} \left\{\int_{\mathbb{D}} \left\{P_0||v||^{\delta}_{\mathbb{C}^n}(w)\right\}^{\frac{2}{\delta}} dA(w)\right\}^{\frac{1}{2}}.
\end{aligned}
$$

As the operator P_0 is L^p bounded for $p > 1$, see [12], this expression will be less than or equal to

$$
\sup_{w\in\mathbb{D}} \left(C \operatorname{tr}\left(B\left((F^*F)^{\frac{2+\epsilon}{2}}\right)(w) B\left((G^*G)^{\frac{2+\epsilon}{2}}\right)(w)\right)\right)^{\frac{1}{2+\epsilon}})||u||_{L^2(\mathbb{C}^n)}||v||_{L^2(\mathbb{C}^n)},
$$

having absorbed the norm of P_0 into the constant C. Estimating the term

$$
\frac{1}{3}\int_{\mathbb{D}} (1-|z|^2)^3 \left\langle T'_{F^*}(u), T'_{G^*}(v)\right\rangle_{\mathbb{C}^n} dA(z)
$$

from the inner product formula is similar.

Finally, let us estimate $3\int_{\mathbb{D}}(1-|z|^2)^2 \langle T_{F^*}(u), T_{G^*}(v)\rangle_{\mathbb{C}^n} \, dA(z)$. We can see from 2.1.2 that

$$\begin{aligned}
&\left|\int_{\mathbb{D}}(1-|z|^2)^2 \left\langle T_{F^*}(u), T_{G^*}(v)\right\rangle_{\mathbb{C}^n} dA(z)\right| \\
&= \left|\int_{\mathbb{D}} \frac{(1-|w|^2)^2}{(1-|w|^2)^2} \int_{\mathbb{D}} \langle Gk_w \otimes Fk_w u, v\rangle_{\mathbb{C}^n} \, dA(z)dA(w)\right| \\
&\leq \left|\int_{\mathbb{D}} ||Gk_w \otimes Fk_w||_{op} dA(w) ||u||_{L^2_a\mathbb{C}^n} ||v||_{L^2_a\mathbb{C}^n}\right| \\
&\leq \sup_{w\in\mathbb{D}} (C \operatorname{tr} B(G^*G)(w)B(F^*F)(w))^{\frac{1}{2}} ||u||_{L^2_a\mathbb{C}^n} ||v||_{L^2_a\mathbb{C}^n}.
\end{aligned}$$

Now we just use Hölder's inequality to get an expression similar to the one in the previous estimate.

$$\begin{aligned}
&\operatorname{tr}(B(G^*G)(w)B(F^*F)(w)) \\
&= \operatorname{tr}(\int_{\mathbb{D}} G(x)^*G(x)|k_w(x)|^2 dA(x) \int_{\mathbb{D}} F(z)^*F(z)|k_w(z)|^2 dA(z)) \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} \operatorname{tr}(G(x)^*G(x)|k_w(x)|^2 F(z)^*F(z)|k_w(z)|^2) dA(x)dA(z) \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} \operatorname{tr}\left\{G(x)^*G(x)F(z)^*F(z)\right\} |k_w(z)|^2|k_w(x)|^2 dA(x)dA(z) \\
&\leq \left\{\int_{\mathbb{D}}\int_{\mathbb{D}} (\operatorname{tr}\left\{G(x)^*G(x)F(z)^*F(z)\right\})^{\frac{2+\epsilon}{2}} |k_w(z)|^2|k_w(x)|^2 dA(x)dA(z)\right\}^{\frac{2}{2+\epsilon}}
\end{aligned}$$

by Hölder. This is then less than or equal to

$$\begin{aligned}
&\left\{\int_{\mathbb{D}}\int_{\mathbb{D}} \left(C \operatorname{tr}\left\{(G(x)^*G(x))^{\frac{2+\epsilon}{2}} (F(z)^*F(z))^{\frac{2+\epsilon}{2}}\right\}\right)\right. \\
&\qquad\qquad \left. \times |k_w(z)|^2|k_w(x)|^2 dA(x)dA(z)\right\}^{\frac{2}{2+\epsilon}}
\end{aligned}$$

by Theorem IX.2.10 on page 258 of [1] and similar steps to before. This final expression is then equal to

$$\begin{aligned}
&\left\{\operatorname{tr}\left(\int_{\mathbb{D}}\int_{\mathbb{D}} ((G(x)^*G(x))^{\frac{2+\epsilon}{2}} (F(z)^*F(z))^{\frac{2+\epsilon}{2}}\right)\right. \\
&\qquad\qquad \left. \times |k_w(z)|^2|k_w(x)|^2 dA(x)dA(z))\right\}^{\frac{2}{2+\epsilon}} \\
&= \left\{\operatorname{tr}(B\left\{(G(x)^*G(x))^{\frac{2+\epsilon}{2}}\right\}(w) B\left\{(F(z)^*F(z))^{\frac{2+\epsilon}{2}}\right\}(w))\right\}^{\frac{2}{2+\epsilon}}.
\end{aligned}$$

Note that here we can use the same reasoning if

$$\int_{\mathbb{D}}\left\{\int_{\mathbb{D}} (C \operatorname{tr}(G(z)F(x)^*F(x)G(z)^*))^{\frac{2+\epsilon}{2}} |k_w(z)|^2 dA(z)\right\} |k_w(x)|^2 dA(x)^{\frac{1}{2+\epsilon}}$$

is uniformly bounded for some $\epsilon > 0$, concluding that our Toeplitz product $T_F T_{G^*}$ will be bounded. This condition is seemingly stronger and less aesthetic than the other one, but it will be used later on when dealing with Toeplitz products that are also invertible. Note that these last inequalities show that the sufficient condition is stronger than the necessary condition.

2.1.2 A Necessary Condition

Proof of Theorem 2.0.3. In [24] (see also [33] for a different approach) Park shows that, for functions f and g in the scalar Bergman space L^2_a, the operator $f \otimes g$ defined by $f \otimes gh = \langle h, g\rangle f$, with $h \in L^2_a$, is equal to

$$T_f T_{\overline{g}} - 2T_z T_f T_{\overline{g}} T_{\overline{z}} + T_z^2 T_f T_{\overline{g}} T_{\overline{z}}^2.$$

Using this result in the vector-valued case, we can see that

$$\begin{aligned} F \otimes G &= \begin{pmatrix} \sum f_{1i} \otimes g_{1i} & \sum f_{1i} \otimes g_{2i} & \cdots \\ \sum f_{2i} \otimes g_{1i} & \ddots & \\ \vdots & & \end{pmatrix} \\ &= T_F T_{G^*} - 2T_z T_F T_{G^*} T_{\overline{z}} + T_z^2 T_F T_{G^*} T_{\overline{z}}^2. \end{aligned}$$

Let us estimate the norm of the operator $(F \circ \phi_w) \otimes (G \circ \phi_w)$, where $F \circ \phi_w$ is the matrix-valued function

$$\begin{pmatrix} f_{11} \circ \phi_w & f_{12} \circ \phi_w & \cdots \\ f_{21} \circ \phi_w & \ddots & \\ \vdots & & \end{pmatrix}.$$

Noting that the operator $(F \circ \phi_w) \otimes (G \circ \phi_w)$ is of finite rank, we can, by Lemma 2.1.3, take as an equivalent norm the square root of the trace of the operator $(F \circ \phi_w \otimes G \circ \phi_w)(G \circ \phi_w \otimes F \circ \phi_w)$. Also by Lemma 2.1.3, we can see that this will be equal to

$$\begin{aligned} &\sum_{q=1}^{n} \sum_{m=1}^{n} (\sum_{r=1}^{n} \sum_{l=1}^{n} \langle f_{qr} \circ \phi_w, f_{ql} \circ \phi_w\rangle_{L^2} \langle g_{ml} \circ \phi_w, g_{mr} \circ \phi_w\rangle_{L^2}) \\ &= \sum_{q=1}^{n} \sum_{m=1}^{n} (\sum_{r=1}^{n} \sum_{l=1}^{n} B(f_{qr}\overline{f_{ql}})(w) B(g_{ml}\overline{g_{mr}})(w)), \end{aligned}$$

which is equal to the trace of the matrix $B(F^*F)(w)B(G^*G)(w)$.

Let U_w be the unitary operator on our vector-valued L^2 space given by $U_w f = (f \circ \phi_w)k_w$. It is well known that $T_{F\circ\phi_w} U_w = U_w T_F$. So, $T_{F\circ\phi_w} = U_w T_F U_w^*$ and thus,

$$\{\mathrm{tr}\,(B(F^*F)(w)B(G^*G)(w))\}^{\frac{1}{2}} \leq C||F \circ \phi_w \otimes G \circ \phi_w||_{op}$$

$$\begin{aligned}
&= C||T_{F\circ\phi_w}T_{G^*\circ\phi_w} - 2T_zT_{F\circ\phi_w}T_{G^*\circ\phi_w}T_{\overline{z}} + T_z^2T_{F\circ\phi_w}T_{G^*\circ\phi_w}T_{\overline{z}}^2||_{op} \\
&= C||U_wT_FU_w^*U_wT_{G^*}U_w^* - 2T_zU_wT_FU_w^*U_wT_{G^*}U_w^*T_{\overline{z}} \\
&\qquad + T_z^2U_wT_FU_w^*U_wT_{G^*}U_w^*T_{\overline{z}}^2||_{op} \\
&= C||(U_wT_FT_{G^*}U_w^* - 2T_zU_wT_FT_{G^*}U_w^*T_{\overline{z}} + T_z^2U_wT_FT_{G^*}U_w^*T_{\overline{z}}^2)||_{op} \\
&= C||(U_wT_FT_{G^*}U_w^* - 2U_wT_{\phi_w}T_FT_{G^*}T_{\overline{\phi_w}}U_w^* + U_wT_{\phi_w}^2T_FT_{G^*}T_{\overline{\phi_w}}^2U_w^*)||_{op}.
\end{aligned}$$

We can now use the triangle inequality on the operator $U_w(T_FT_{G^*} - 2T_{\phi_w}T_FT_{G^*}T_{\overline{\phi_w}} + T_{\phi_w}^2T_FT_{G^*}T_{\overline{\phi_w}}^2)U_w^*$, as in [33], along with the fact that $||T_{\phi_w}|| \leq 1$ to get our result. $\square$

Chapter 3

Bounded and Invertible Toeplitz Products

In the following we will be working with square matrices F and G with entries from the scalar-valued Bergman space $L^2_a(\mathbb{D})$. Where it is not explicitly stated otherwise, this will be the case.

3.1 Bounded and Invertible Toeplitz Products

3.1.1 Main Theorem

The main theorem of this chapter is a characterization of bounded and invertible Toeplitz products in the vector case. The theory of matrix A_2 weights will play an important role here.

Theorem 3.1.1.

The Toeplitz product $T_F T_{G^*}$ is bounded and invertible if and only if

$$\sup_{w\in\mathbb{D}} ||B(F^*F)(w)B(G^*G)(w)|| < \infty,$$

and there exists $\eta > 0$ such that $(FG^*GF^*)(z) \geq \eta I$ for all $z \in \mathbb{D}$. This last inequality is a matrix inequality.

3.1.2 A Reverse Hölder Inequality

We will now develop some of the theory needed to show a reverse Hölder inequality used to characterize the matrices of analytic functions, F and G, such that the Toeplitz product

$T_F T_{G^*}$ is bounded and invertible on the vector-valued Bergman space. Compare this next lemma with Lemma 4.3 in [37].

Lemma 3.1.2. *Let $D(w, s)$ be the pseudohyperbolic disk with radius $0 < s < 1$ and centre w. If $F(z)$ is invertible for all $z \in \mathbb{D}$, then for $z \in D(w, s)$ there exists a constant, η_s, dependent only on s such that*

$$(F^*(w)F(w)) \leq B(F^*F)(w),$$

$$(F^*(z)F(z)) \leq \eta_s B(F^*F)(w),$$

$$(F^{-1}(w)F^{*-1}(w)) \leq B(F^{-1}F^{*-1})(w)$$

and

$$(F^{-1}(z)F^{-1*}(z)) \leq \eta_s B(F^{-1}F^{*-1})(w).$$

Proof. Let $\mathbf{e}$ be an arbitrary vector. For $F \in L^2_a(\mathbb{C}^n)$,

$$\begin{aligned}
\langle F(u)\mathbf{e}, F(u)\mathbf{e}\rangle &= \left\langle \int F(z)\overline{K_u(z)}dA(z)\mathbf{e}, \int F(z)\overline{K_u(z)}dA(z)\mathbf{e} \right\rangle \\
&= || \int F(z)\overline{K_u(z)}dA(z)\mathbf{e}||^2_{\mathbb{C}^n} \\
&\leq \int \langle F\mathbf{e}, F\mathbf{e}\rangle \, dA(z)||K_u||^2_{L^2} = \left\langle \int F^*(z)F(z)dA(z)\mathbf{e}, \mathbf{e} \right\rangle ||K_u||^2_{L^2}.
\end{aligned}$$

So, if $u \in D(0, s)$, then

$$F^*(u)F(u) \leq \int F^*(z)F(z)dA(z)||K_u||^2_2 \leq \int F^*(z)F(z)dA(z)\frac{1}{(1-s^2)^2}.$$

If $z \in D(w, s)$, then $z = \phi_w(u)$ for some $u \in D(0, s)$. Thus,

$$\begin{aligned}
F^*(z)F(z) &= F^*(\phi_w(u))F(\phi_w(u)) \\
&\leq \int F^*(\phi_w(z))F(\phi_w(z))dA(z)\frac{1}{(1-s^2)^2} \\
&= B(F^*F)(w)\frac{1}{(1-s^2)^2}.
\end{aligned}$$

This proves the second inequality.

Now let us show that $F^{-1}(w)F^{*-1}(w) \leq B(F^{-1}F^{*-1})(w)$:

$$\begin{aligned}
\left\langle F^{*-1}(w)\mathbf{e}, F^{*-1}(w)\mathbf{e}\right\rangle &= \left\langle F^{*-1}(\phi_w(0))\mathbf{e}, F^{*-1}(\phi_w(0))\mathbf{e}\right\rangle \\
&= \left\langle \int F^{-1}(\phi_w(z))F^{*-1}(\phi_w(z))dA(z)\mathbf{e}, \mathbf{e} \right\rangle,
\end{aligned}$$

and we arrive at the conclusion that $F^{-1}(w)F^{*-1}(w) \leq B(F^{-1}F^{*-1})(w)$ in a similar manner to before.

So for $z \in D(w,s)$ we know that $F^{-1}(w)F^{*-1}(w) \leq B(F^{-1}F^{*-1})(w)$ and $F^*(z)F(z) \leq B(F^*F)(w)\frac{1}{(1-s^2)^2}$. The other inequalities follow from applying the same procedure to $F^{-1}F^{*-1}$ instead of F^*F.

$\square$

Lemma 3.1.3. *If there exists η such that $F(z)G(z)^*G(z)F(z)^* > \eta I$ for all $z \in \mathbb{D}$ and* $\mathrm{tr}(B(G^*G)(w)B(F^*F)(w))$ *is uniformly bounded on $\mathbb{D}$, then*

$$||B(F^{-1}(F^*)^{-1})(w)^{\frac{1}{2}}B(F^*F)(w)^{\frac{1}{2}}||$$

is uniformly bounded on $\mathbb{D}$.

Proof. Let us suppose that $F(w)G(w)^*G(w)F(w)^* > \eta I$ for all $w \in \mathbb{D}$. Then $B(G^*G)(w) \geq G(w)^*G(w) \geq \eta(F(w)^*F(w))^{-1}$. The key inequality here is $G(w)^*G(w) \geq \eta(F(w)^*F(w))^{-1}$ as this implies that $B(G^*G)(w) \geq \eta B((F^*F)^{-1})(w)$, and so

$$\begin{aligned}&(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}}\\ &\geq \eta(B(F^*F)(w))^{\frac{1}{2}}B((F^*F)^{-1}(w))(B(F^*F)(w))^{\frac{1}{2}}.\end{aligned}$$

Thus, as $||(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}}|| < M$ for all w and some constant $M > 0$,

$$\begin{aligned}&\mathrm{tr}(B(F^{-1}(F^*)^{-1})(w)B(F^*F)(w))\\ &\leq \frac{1}{\eta}C||(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}}|| < \eta^{-1}CM.\end{aligned}$$

And so,

$$\begin{aligned}&||B(F^{-1}(F^*)^{-1})(w))^{\frac{1}{2}}B(F^*F)(w)^{\frac{1}{2}}||^2\\ &= ||(B(F^{-1}(F^*)^{-1})(w))^{\frac{1}{2}}B(F^*F)(w)(B(F^{-1}(F^*)^{-1})(w))^{\frac{1}{2}}||\\ &\leq C\,\mathrm{tr}(B(F^{-1}(F^*)^{-1})(w)B(F^*F)(w))\\ &\leq \frac{1}{\eta}C||(B(F^*F)(w))^{\frac{1}{2}}B(G^*G)(w)(B(F^*F)(w))^{\frac{1}{2}}||,\end{aligned}$$

where C and η are constants independent of w, and C may change from line to line.

$\square$

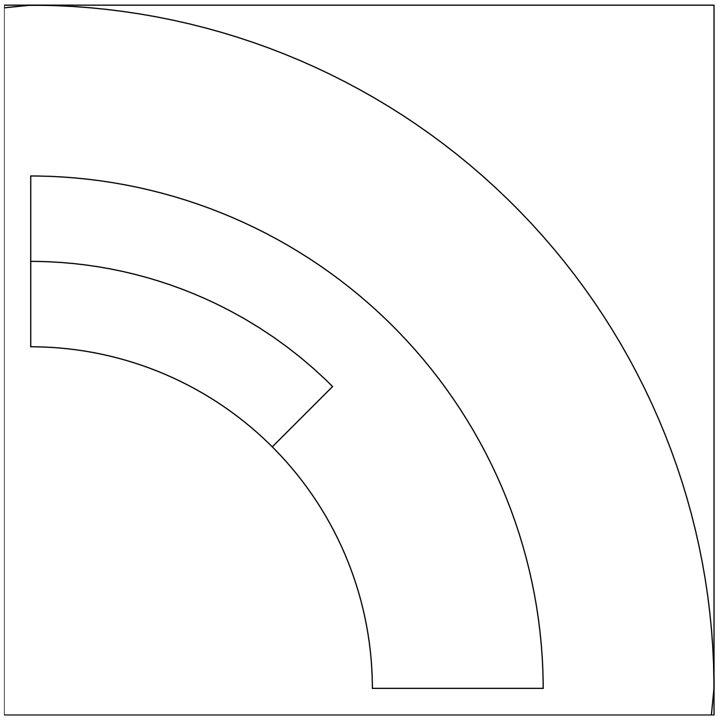

Figure 3.1: Two nested dyadic rectangles in the unit disk.

Definition 3.1.4. A dyadic rectangle $Q_{j,k,l}$ is a subset of the unit disk of the form

$$\left\{z = re^{i\theta} : (k-1)2^{-j} \leq r \leq k2^{-j}, (l-1)2^{1-j}\pi \leq \theta \leq l2^{1-j}\pi\right\},$$

where $j \geq 0$ and $1 \leq k, l \leq 2^j$.

Lemma 3.1.5. *There exists $0 < r < 1$ such that for all dyadic rectangles Q with positive distance to the boundary, $\partial\mathbb{D}$, $Q \subset D(z_Q, r)$. Here, D is the pseudohyperbolic disk and z_Q is the centre of the dyadic rectangle Q.*

Proof. This is just Proposition 4.4 in [37]. □

Compare this next lemma with Lemma 4.5 in [37].

Lemma 3.1.6. *If*

$$\sup_{w \in \mathbb{D}} ||B(F^{-1}(F^*)^{-1})(w))^{\frac{1}{2}} B(F^*F)(w)^{\frac{1}{2}}|| < \infty,$$

then

$$\sup_{Q:dyadic} ||\left\{\frac{1}{|Q|}\int_Q (F^*F)dA(z)\right\}^{\frac{1}{2}} \left\{\frac{1}{|Q|}\int_Q (F^{-1}F^{*-1})dA(z)\right\}^{\frac{1}{2}}|| < \infty.$$

Comparing our proof to the proof of Lemma 4.5 in [37] highlights some of the obstacles that noncommutativity causes us.

Proof. If the dyadic rectangle Q is the whole disk, then as

$$\int_{\mathbb{D}} F^* F dA(z) = B(F^* F)(0)$$

and

$$\int_{\mathbb{D}} F^{-1} F^{*-1} dA(z) = B(F^{-1} F^{*-1})(0),$$

we see that

$$\begin{aligned} &|| \left(\int_{\mathbb{D}} F^* F dA(z) \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} F^{-1} F^{*-1} dA(z) \right)^{\frac{1}{2}} || \\ &= ||B(F^{-1}(F^*)^{-1})(0)^{\frac{1}{2}} B(F^* F)(0)^{\frac{1}{2}}||. \end{aligned}$$

Now let us suppose that our dyadic rectangle Q has a positive distance from the boundary. By Lemma 3.1.5 our rectangle Q will be strictly contained in a pseudohyperbolic disk $D(z_Q, R)$, z_Q being the centre of our dyadic rectangle and R being the same for each dyadic rectangle. Thus, by Lemma 3.1.2,

$$(F^{-1}(z) F^{-1*}(z)) \leq \eta B(F^{-1} F^{*-1})(z_Q)$$

and

$$(F^*(z) F(z)) \leq \eta B(F^* F)(z_Q)$$

for all z in our pseudohyperbolic disk $D(z_Q, R)$. Here the constant η will only be dependent on R, which is the same for all of these dyadic rectangles.

Thus, using the fact that if A, B and C are positive matrices such that $A \leq B$, then $C^{\frac{1}{2}} A C^{\frac{1}{2}} \leq C^{\frac{1}{2}} B C^{\frac{1}{2}}$ and $\operatorname{tr}(C^{\frac{1}{2}} A C^{\frac{1}{2}}) \leq \operatorname{tr}(C^{\frac{1}{2}} B C^{\frac{1}{2}})$, we can deduce the following series of inequalities from our hypothesis:

$$|| \left\{ \frac{1}{|Q|} \int_Q (F^* F) dA(z) \right\}^{\frac{1}{2}} \left\{ \frac{1}{|Q|} \int_Q (F^{-1} F^{*-1}) dA(z) \right\}^{\frac{1}{2}} ||^2$$

$$
\begin{aligned}
&= \left|\left| \left\{ \frac{1}{|Q|} \int_Q (F^*F) dA(z) \right\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^{-1}F^{*-1}) dA(z) \right.\right. \\
&\qquad\qquad \left.\left. \times \left\{ \frac{1}{|Q|} \int_Q (F^*F) dA(z) \right\}^{\frac{1}{2}} \right|\right| \\
&\leq \operatorname{tr}\left(\left\{ \frac{1}{|Q|} \int_Q (F^*F) dA(z) \right\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^*F)^{-1} dA(z) \right. \\
&\qquad\qquad \left. \times \left\{ \frac{1}{|Q|} \int_Q (F^*F) dA(z) \right\}^{\frac{1}{2}} \right) \\
&\leq \operatorname{tr}\left(\left\{ \frac{1}{|Q|} \int_Q (F^*F) dA(z) \right\}^{\frac{1}{2}} \{\eta B(F^{-1}F^{*-1})(z_Q)\} \right. \\
&\qquad\qquad \left. \times \left\{ \frac{1}{|Q|} \int_Q (F^*F) dA(z) \right\}^{\frac{1}{2}} \right) \\
&= \operatorname{tr}\Big(\{\eta(B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \frac{1}{|Q|} \int_Q (F^*F) dA(z) \\
&\qquad\qquad \times \{\eta(B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \Big) \\
&\leq \operatorname{tr}\Big(\{\eta(B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \{\eta B(F^*F)(z_Q)\} \\
&\qquad\qquad \times \{(\eta B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \Big) \\
&\leq C||\Big(\{\eta(B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \{\eta B(F^*F)(z_Q)\} \\
&\qquad\qquad \times \{\eta(B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \Big)|| \\
&\leq C\eta^2|| \{(B(F^{-1}F^{*-1})(z_Q))\}^{\frac{1}{2}} \{B(F^*F)(z_Q)\}^{\frac{1}{2}} ||^2 < M.
\end{aligned}
$$

Note that C is a constant that possibly changes from line to line and is dependent on the dimension of $\mathbb{C}^n$ only. M will be dependent only on the uniform bound of $||B(F^{-1}F^{*-1})^{\frac{1}{2}}(w)B(F^*F)(w)^{\frac{1}{2}}||$, the dimension we are working in and the constant R, which is the same for each dyadic rectangle not touching the boundary.

What happens when we have a dyadic rectangle that touches the boundary but is not the whole disk? We can see that the centre of the rectangle z_Q is at a distance of at least $1/2$ from the centre, i.e. $|z_Q| \geq 1/2$. Then

$$
\begin{aligned}
B(F^*F)(z_Q) &= \int_{\mathbb{D}} F^*(z)F(z)|k_{z_Q}|^2(z) dA(z) \\
\geq \int_Q F^*(z)F(z)|k_{z_Q}(z)|^2 dA(z) &\geq \frac{c}{(1-|z_Q|)^2} \int_Q F^*(z)F(z) dA(z)
\end{aligned}
$$

by Lemma 4.2 in [37]. We can also see that in this case $|Q| = 8|z_Q|(1-|z_Q|)^2$, and so

$$
B(F^*F)(z_Q) \geq \frac{4c}{|Q|} \int_Q F^*(z)F(z) dA(z).
$$

We can do the same for $F^{-1}F^{*-1}$ to see that

$$B(F^{-1}F^{*-1})(z_Q) \geq \frac{4c}{|Q|}\int_Q F^{-1}(z)F^{*-1}(z)dA(z).$$

We can then combine these and take the trace to see that

$$\begin{aligned}
&\operatorname{tr}\left(\frac{1}{|Q|^2}\left\{\int_Q F^*F(z)dA(z)\right\}^{\frac{1}{2}}\int_Q (F^*F)^{-1}(z)dA(z)\left\{\int_Q F^*F(z)dA(z)\right\}^{\frac{1}{2}}\right)\\
&\leq (4c)^{-1}\operatorname{tr}\Big(\left\{B(F^{-1}F^{*-1})(z_Q)\right\}^{\frac{1}{2}}\frac{1}{|Q|}\int_Q F^*(z)F(z)dA(z)\\
&\qquad\qquad\qquad\qquad \times\left\{B(F^{-1}F^{*-1})(z_Q)\right\}^{\frac{1}{2}}\Big)\\
&\leq (16c^2)^{-1}\operatorname{tr}\left(\left\{B(F^{-1}F^{*-1})(z_Q)\right\}^{\frac{1}{2}}B(F^*F)(z_Q)\left\{B(F^{-1}F^{*-1})(z_Q)\right\}^{\frac{1}{2}}\right)\\
&\leq C||\left\{B(F^{-1}F^{*-1})(z_Q)\right\}^{\frac{1}{2}}\left\{B(F^*F)(z_Q)\right\}^{\frac{1}{2}})||^{\frac{1}{2}} < M',
\end{aligned}$$

where M' is independent of Q.

□

If $||\left\{\frac{1}{|Q|}\int_Q(F^*F)dA(z)\right\}^{\frac{1}{2}}\left\{\frac{1}{|Q|}\int_Q(F^{-1}F^{*-1})dA(z)\right\}^{\frac{1}{2}}|| < M$ for all dyadic rectangles Q and some constant M, we will say that F^*F has the matrix A_2 condition. See [38] for a similar notion of matrix weights. We will now find a characterization of such functions F in terms of the boundedness of certain averaging operators on the function space $L^2(F^*F)$.

Theorem 3.1.7. *If, for $F \in M_{n\times n}(L^2_a)$, the matrix F^*F has the A_2 condition, then the dyadic averaging operators, $f \mapsto \chi_Q\frac{1}{|Q|}\int_Q f(z)dA(z)$ for dyadic Q, are uniformly bounded on a dense subset, $L^2(\mathbb{C}^n)\cap L^2(F^*F)$, of $L^2(F^*F)$.*

The proofs here and in the next theorem follow the reasoning in Lemma 2.1 in [38].

Proof. For a given dyadic rectangle Q, let R be the subspace $\{\chi_Q\frac{1}{|Q|^{\frac{1}{2}}}\mathbf{e} : \mathbf{e} \in \mathbb{C}^n\}$. We can see that the orthogonal projection from $L^2(\mathbb{D}, \mathbb{C}^n)$ onto R is given by

$$P_Q : f \mapsto \chi_Q\frac{1}{|Q|}\int_Q f(z)dA(z).$$

So, we wish to show that these projections are uniformly bounded with respect to the $L^2(F^*F)$ norm. Clearly,

$$||P_Q||_{L^2(F^*F)} = \sup_{\{f\in L^2\cap L^2(F^*F):||f||_{L^2(F^*F)}\neq 0\}}\left\{\frac{||P_Qf||_{L^2(F^*F)}}{||f||_{L^2(F^*F)}}\right\}.$$

If we let S denote the orthogonal complement of R in L^2, then $f = f_1 + f_2$, where $f_1 \in R$ and $f_2 \in S' = S \cap L^2(F^*F)$. Thus the expression for the norm of the projection will become

$$\sup_{\{f_1+f_2 \in L^2 \cap L^2(F^*F): ||f||_{L^2(F^*F)} \neq 0\}} \left\{ \frac{||f_1||_{L^2(F^*F)}}{||f_1+f_2||_{L^2(F^*F)}} \right\}$$
$$= \sup_{\{\mathbf{e} \in \mathbb{C}^n : \mathbf{e} \neq 0\}} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}||_{L^2(F^*F)}}{\inf_{\{f_2 \in S\}} ||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e} + f_2||_{L^2(F^*F)}} \right\}$$
$$= \sup_{\{\mathbf{e} \in \mathbb{C}^n : \mathbf{e} \neq 0\}} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}||_{L^2(F^*F)}}{\mathrm{dist}_{L^2(F^*F)} \left(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, S' \right)} \right\}.$$

Let us look at $\mathrm{dist}_{L^2(F^*F)} \left(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, S' \right)$.

$$\mathrm{dist}_{L^2(F^*F)} \left(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, S' \right) = \mathrm{dist}_{L^2} \left((F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, (F^*F)^{\frac{1}{2}} S' \right)$$
$$= \sup_{\left\{ h \in \left((F^*F)^{\frac{1}{2}} S' \right)^{\perp} : ||h|| = 1 \right\}} \left\langle (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, h \right\rangle,$$

$(F^*F)^{-1}$ exists as we have the A_2 condition. Note that

$$\left((F^*F)^{\frac{1}{2}} S' \right)^{\perp} = \left((F^*F)^{-\frac{1}{2}} R \right).$$

Then we can see that:

$$\mathrm{dist}_{L^2(F^*F)} \left(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, S' \right) = \sup_{\left\{ h \in \left((F^*F)^{-\frac{1}{2}} R \right) : ||h|| = 1 \right\}} \left\langle (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, h \right\rangle$$
$$= \sup_{\left\{ \mathbf{g} \in \mathbb{C}^n : ||(F^*F)^{-\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{g}|| \leq 1 \right\}} \left\langle (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, (F^*F)^{-\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{g} \right\rangle$$
$$= \sup_{\left\{ \mathbf{g} \in \mathbb{C}^n : \frac{1}{|Q|} \int_Q \langle (F^*F)^{-1} \mathbf{g}, \mathbf{g} \rangle \leq 1 \right\}} \left\langle (F^*F)^{\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, (F^*F)^{-\frac{1}{2}} \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{g} \right\rangle$$
$$= \sup_{\left\{ \mathbf{g} \in \mathbb{C}^n : \frac{1}{|Q|} \int_Q \langle (F^*F)^{-1} \mathbf{g}, \mathbf{g} \rangle \leq 1 \right\}} \left\langle \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{e}, \chi_Q \frac{1}{|Q|^{\frac{1}{2}}} \mathbf{g} \right\rangle$$
$$= \sup_{\{\mathbf{h} \in \mathbb{C}^n : ||\mathbf{h}|| \leq 1\}} \left\langle \mathbf{e}, \left\{ \int_{\mathbb{D}} F^{-1} F^{*-1} \chi_Q \frac{1}{|Q|} \right\}^{-\frac{1}{2}} \mathbf{h} \right\rangle$$
$$= || \left\{ \frac{1}{|Q|} \int_Q F^{-1} F^{*-1} \right\}^{-\frac{1}{2}} \mathbf{e} ||.$$

Let us now put this equivalent expression for the distance back into our expression for the norm of the projection in $L^2(F^*F)$:

$$\begin{aligned}
||P_Q||_{L^2(F^*F)\to L^2(F^*F)} &= \sup_{\{\mathbf{e}\in\mathbb{C}^n:\mathbf{e}\neq 0\}} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}\mathbf{e}||_{L^2(F^*F)}}{\mathrm{dist}_{L^2(F^*F)}\left(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}\mathbf{e}, S'\right)} \right\} \\
&= \sup_{\{\mathbf{e}\in\mathbb{C}^n:\mathbf{e}\neq 0\}} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}\mathbf{e}||_{L^2(F^*F)}}{||\left\{\frac{1}{|Q|}\int_Q F^{-1}F^{*-1}\right\}^{-\frac{1}{2}}\mathbf{e}||} \right\} \\
&= \sup_{\{\mathbf{e}\in\mathbb{C}^n:\mathbf{e}\neq 0\}} \left\{ \frac{||\left\{\frac{1}{|Q|}\int_Q F^*F\right\}^{\frac{1}{2}}\mathbf{e}||}{||\left\{\frac{1}{|Q|}\int_Q F^{-1}F^{*-1}\right\}^{-\frac{1}{2}}\mathbf{e}||} \right\} \\
&= \sup_{\{\mathbf{e}\in\mathbb{C}^n:\mathbf{e}\neq 0\}} \left\{ \frac{||\left\{\frac{1}{|Q|}\int_Q F^*F\right\}^{\frac{1}{2}}\mathbf{e}||}{||\left\{\frac{1}{|Q|}\int_Q F^{-1}F^{*-1}\right\}^{-\frac{1}{2}}\mathbf{e}||} \right\} \\
&= ||\left\{\frac{1}{|Q|}\int_Q F^*F\right\}^{\frac{1}{2}}\left\{\frac{1}{|Q|}\int_Q F^{-1}F^{*-1}\right\}^{\frac{1}{2}}||.
\end{aligned}$$

$\square$

Lemma 3.1.8. *If the averaging operators $g \mapsto \chi_Q \frac{1}{|Q|}\int_Q g(z)dA(z)$ are uniformly bounded on $L^2(|f|^2)$ for all dyadic rectangles Q, then $|f|^2$ has the scalar A_2 condition.*

Proof. Again we can see that the averaging operator $g \mapsto \chi_Q \frac{1}{|Q|}\int_Q g(z)dA(z)$ is the projection $P : L^2 \to \chi_Q \frac{1}{|Q|^{\frac{1}{2}}}\mathbb{C}$. We are working, as before, on the dense subset $L^2(\mathbb{C}) \cap L^2(|f|^2)$. If we assume that $\frac{1}{|f|^2}$ is bounded, then we can, as before, show that

$$\begin{aligned}
&\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}z, S') \\
&= \left| \left[\int_{\mathbb{D}} \frac{1}{|f|^2}\chi_Q \frac{1}{|Q|}\right]^{-\frac{1}{2}} z \right| = \left| \left[\int_{\mathbb{D}} \frac{1}{|f|^2}\chi_Q \frac{1}{|Q|}\right]^{-\frac{1}{2}} \right|,
\end{aligned}$$

where $|z| = 1$. Now, if we drop this assumption on $\frac{1}{|f|^2}$ but instead use $\frac{1}{|f|^2+\epsilon}$ for $\epsilon > 0$, then we can see that

$$\begin{aligned}
\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}z, S') &= \lim_{\epsilon\to 0} \mathrm{dist}_{L^2(|f|^2+\epsilon)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}z, S') \\
&= \lim_{\epsilon\to 0} \left| \left[\frac{1}{|Q|}\int_Q \frac{1}{|f|^2+\epsilon}\right]^{-\frac{1}{2}} \right|,
\end{aligned}$$

where S' is the intersection of the orthogonal complement of $\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}\mathbb{C}$ with $L^2(|f|^2)$, and $|z| = 1$. As the norm of our bounded projection P is

$$\begin{aligned}
&\sup_{z\in\mathbb{C}:z\neq 0,|z|=1} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z||_{L^2(|f|^2)}}{\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z, S')} \right\} \\
&= \sup_{z\in\mathbb{C}:z\neq 0,|z|=1} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}}||_{L^2(|f|^2)}}{\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z, S')} \right\},
\end{aligned}$$

we know that $\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z, S')$ is nonzero for nonzero z and hence

$$\lim_{\epsilon\to 0} \left| \left[\frac{1}{|Q|} \int_Q \frac{1}{|f|^2 + \epsilon} \right] \right| < \infty,$$

and so by the Monotone Convergence Theorem

$$\left| \left[\frac{1}{|Q|} \int_Q \frac{1}{|f|^2} \right] \right| < \infty,$$

and

$$\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z, S') = \left| \left[\frac{1}{|Q|} \int_Q \frac{1}{|f|^2} \right]^{-\frac{1}{2}} \right|,$$

where $|z| = 1$. Thus

$$\begin{aligned}
||P||_{L^2(|f|^2)} &= \sup_{z\in\mathbb{C}:z\neq 0,|z|=1} \left\{ \frac{||\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z||_{L^2(|f|^2)}}{\mathrm{dist}_{L^2(|f|^2)}(\chi_Q \frac{1}{|Q|^{\frac{1}{2}}} z, S')} \right\} \\
&= \sup_{z\in\mathbb{C}:z\neq 0,|z|=1} \left\{ \frac{\left| \left[\frac{1}{|Q|} \int_Q |f|^2 \right]^{\frac{1}{2}} \right|}{\left| \left[\frac{1}{|Q|} \int_Q \frac{1}{|f|^2} \right]^{-\frac{1}{2}} \right|} \right\} \\
&= \left\{ \left| \left[\frac{1}{|Q|} \int_Q |f|^2 \right]^{\frac{1}{2}} \right| \left| \left[\frac{1}{|Q|} \int_Q \frac{1}{|f|^2} \right]^{\frac{1}{2}} \right| \right\},
\end{aligned}$$

which is uniformly bounded as required.

□

Compare the next lemma with Lemma 3.6 in [38].

Lemma 3.1.9. *If F^*F has the A_2 condition, then* $\mathrm{tr}(F^*F)$ *has the scalar A_2 condition.*

Proof. We will show that each element on the diagonal of F^*F has the scalar A_2 condition. We can then deduce that the sum of these will also have the A_2 condition. Firstly we know that if F^*F has the A_2 condition, then the operators $f \mapsto \chi_Q \frac{1}{|Q|} \int_Q f(z)dA(z)$ are uniformly bounded on $L^2(F^*F)$ for $f \in L^2(\mathbb{C}^n)$. So let us take $g \in L^2(\mathbb{D}) \cap L^2(\mathbb{D}\langle F^*Fe_j, e_j\rangle)$, where $\langle F^*Fe_j, e_j\rangle$ is the scalar-valued function $z \mapsto \langle F^*(z)F(z)e_j, e_j\rangle$. Then note that $ge_j \mapsto \chi_Q \frac{1}{|Q|}\int_Q g(z)e_j dA(z)$ is uniformly bounded on $L^2_a(\mathbb{D}, \mathbb{C}^n)$. This implies that $g \mapsto \chi_Q \frac{1}{|Q|}\int_Q g(z)dA(z)$ is uniformly bounded with respect to the scalar measure $\langle F^*Fe_j, e_j\rangle_{\mathbb{C}^n}$, which will be whatever diagonal element of F^*F we want. Thus by the previous lemma the trace of F^*F will have the scalar A_2 condition.

□

Compare this next lemma with Lemma 4.6 in [37], Lemma 2.5 in [34] and also 1.7 on page 196 of [32].

Lemma 3.1.10. *If a scalar-valued function $|f|^2$ has the A_2 condition and if $0 < \delta < 1$, then for each dyadic rectangle Q and $E \subset Q$ such that $|E| \leq \delta|Q|$ we have that $\mu(E) \leq \lambda\mu(Q)$ for some $0 < \lambda < 1$, where $d\mu = |f|^2 dA$ and λ only depends on δ and the A_2 constant of $|f|^2$.*

Proof.

$$\begin{aligned} |Q \setminus E|^2 = \left\{\int_{Q/E} |f||f|^{-1} dA\right\}^2 &\leq \left\{\int_{Q\setminus E} |f|^2 dA\right\}\left\{\int_{Q\setminus E} |f|^{-2} dA\right\} \\ &\leq \left\{\int_{Q\setminus E} |f|^2 dA\right\}\left\{\int_{Q} |f|^{-2} dA\right\} \\ &\leq \left\{\int_{Q\setminus E} |f|^2 dA\right\} C|Q|^2 \left\{\int_{Q} |f|^2 dA\right\}^{-1}, \end{aligned}$$

by our A_2 condition on $|f|^2$. This equals

$$\begin{aligned} &(\left\{\int_Q |f|^2 dA\right\} - \left\{\int_E |f|^2 dA\right\})C|Q|^2 \left\{\int_Q |f|^2 dA\right\}^{-1} \\ &= C\left(1 - \left\{\int_E |f|^2 dA\right\}\left\{\int_Q |f|^2 dA\right\}^{-1}\right)|Q|^2, \end{aligned}$$

so we know that

$$|Q \setminus E|^2 \leq C|Q|^2\left(1 - \frac{\mu(E)}{\mu(Q)}\right),$$

and thus

$$\frac{|Q \setminus E|^2}{|Q|^2} \leq C\frac{\mu(Q \setminus E)}{\mu(Q)}.$$

From our hypothesis we know that $\frac{|E|}{|Q|} \leq \delta < 1$, this implies that $\frac{|Q \setminus E|}{|Q|} \geq 1 - \delta > 0$. So we can now deduce that

$$0 < \frac{(1-\delta)^2}{C} \leq \frac{1}{C}\frac{|Q \setminus E|^2}{|Q|^2} \leq \frac{\mu(Q \setminus E)}{\mu(Q)}.$$

This lets us now see that

$$1 = \frac{\mu(Q)}{\mu(Q)} = \frac{\mu(Q \setminus E) + \mu(E)}{\mu(Q)} \geq \frac{\mu(E)}{\mu(Q)} + \frac{(1-\delta)^2}{C},$$

and hence

$$\frac{\mu(E)}{\mu(Q)} \leq 1 - \frac{(1-\delta)^2}{C}.$$

□

The following lemma will be crucial to our application of the A_2 condition.

Lemma 3.1.11. *If F^*F has the A_2 condition and J is a strictly positive matrix, then JF^*FJ will have the A_2 condition. The A_2 constant of JF^*FJ will depend on the A_2 bound of F^*F and the dimension only.*

Proof.

$$\begin{aligned}
&\left\|\left(\frac{1}{|I|}\int_I JF^*FJ\right)^{\frac{1}{2}}\left(\frac{1}{|I|}\int_I (JF^*FJ)^{-1}\right)^{\frac{1}{2}}\right\|^2 \\
&= \left\|\left(\frac{1}{|I|}\int_I JF^*FJ\right)^{\frac{1}{2}}\left(\frac{1}{|I|}\int_I (JF^*FJ)^{-1}\right)\left(\frac{1}{|I|}\int_I JF^*FJ\right)^{\frac{1}{2}}\right\| \\
&\leq C\,\mathrm{tr}\left(\left(\frac{1}{|I|}\int_I JF^*FJ\right)^{\frac{1}{2}}\left(\frac{1}{|I|}\int_I (JF^*FJ)^{-1}\right)\left(\frac{1}{|I|}\int_I JF^*FJ\right)^{\frac{1}{2}}\right),
\end{aligned}$$

where the constant C depends only on the dimension. This equals

$$\begin{aligned}
&C\,\mathrm{tr}\left(\left(\frac{1}{|I|}\int_I (F^*F)^{-1}\right)\left(\frac{1}{|I|}\int_I F^*F\right)\right) \\
&\leq C'\left\|\left(\frac{1}{|I|}\int_I F^*F\right)^{\frac{1}{2}}\left(\frac{1}{|I|}\int_I (F^*F)^{-1}\right)^{\frac{1}{2}}\right\|^2.
\end{aligned}$$

Again C' depends only on the dimension, thus giving us our result.

□

Definition 3.1.12. The dyadic maximal operator M_Δ is defined by

$$(M_\Delta f)(w) = \sup_{w \in Q} \frac{1}{|Q|} \int_Q |f(z)| dA(z),$$

where the Q are dyadic rectangles and $f \in L^2$.

Theorem 3.1.13. *(The Calderon-Zygmund Decomposition Theorem.) Let $f \in L^1(\mathbb{D})$. If we have $t > 0$ such that the set $\Lambda = \{z \in \mathbb{D} : M_\Delta f(z) > t\}$ is not the whole of $\mathbb{D}$, then we can decompose Λ into a disjoint union of dyadic rectangles Q_i such that $t < \frac{1}{|Q_i|} \int_{Q_i} |f(z)| dA(z) < 8t$.*

Proof. The proof of this is exactly as in [34] and [37]. □

Compare this next lemma with Proposition 4.11 in [37].

Lemma 3.1.14. *The trace of F^*F satisfies the following:*

1.
$$\operatorname{tr}(F^*F) \leq M_\Delta \operatorname{tr}(F^*F)$$

on $\mathbb{D}$ and

2.
$$\int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z)) dA(z) \leq M_\Delta \operatorname{tr}(F^*F)(0) \leq (4/3)^2 \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z)) dA(z).$$

Proof. 1. This follows from Proposition 4.11 in [37]. We just need to note that $\operatorname{tr}(F^*F)$ is continuous and the proof works as it is.

2. $\mathbb{D}$ is a dyadic rectangle containing 0 so

$$M_\Delta \operatorname{tr}(F^*F)(0) \geq \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z)) dA(z) = \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z)) dA(z).$$

Let us take a dyadic rectangle Q containing 0 which is not the unit disk. We know that Q will be contained in the pseudohyperbolic disk $D(0, \frac{1}{2})$. Let $\mathbf{e} \in \mathbb{C}^n$, then as $F\mathbf{e} \in L^2_a(\mathbb{C}^n)$,

$$\begin{aligned}
\langle F(u)\mathbf{e}, F(u)\mathbf{e} \rangle = ||F(u)\mathbf{e}||^2_{\mathbb{C}^n} &= || \int F(z)\overline{K_u(z)} dA(z)\mathbf{e}||^2_{\mathbb{C}^n} \\
&\leq \left\{ \int ||F(z)\mathbf{e}|| |K_u(z)| dA(z) \right\}^2 \\
&\leq \int \langle F\mathbf{e}, F\mathbf{e} \rangle \, dA(z) ||K_u(z)||^2_{L^2} \\
&= \left\langle \int F^*(z)F(z) dA(z)\mathbf{e}, \mathbf{e} \right\rangle ||K_u(z)||^2_{L^2}.
\end{aligned}$$

So,

$$\begin{aligned} F^*(u)F(u) &\leq \int F^*(z)F(z)dA(z)||K_u||_2^2 \\ &\leq \int F^*(x)F(x)dA(x)\frac{1}{(1-\frac{1}{2}^2)^2} \\ &= \left(\frac{4}{3}\right)^2 \int F^*(x)F(x)dA(x) \end{aligned}$$

on each Q containing 0 which is not $\mathbb{D}$. So,

$$\operatorname{tr}(F^*(u)F(u)) \leq \operatorname{tr}\left(\left(\frac{4}{3}\right)^2 \int_{\mathbb{D}} F^*(x)F(x)dA(x)\right)$$

for $u \in Q$. Hence

$$\frac{1}{|Q|}\int_Q \operatorname{tr}(F^*(z)F(z))dA(z) \leq \left(\frac{4}{3}\right)^2 \int_{\mathbb{D}} \operatorname{tr}(F^*(x)F(x))dA(x),$$

and so

$$M_\Delta \operatorname{tr}(F^*F)(0) \leq (4/3)^2 \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z))dA(z).$$

□

The proof of the following theorem follows the logic of Theorem 2.1 in [34] and Theorem 4.1 in [37]. It contains the key to the proof of Theorem 3.1.1, i.e. the reverse Hölder property.

Theorem 3.1.15. *If F^*F satisfies A_2, then there exists $\epsilon > 0$ such that*

$$\int (\operatorname{tr}(F^*(z)F(z)))^{1+\epsilon} dA(z) \leq C\left(\int \operatorname{tr}(F^*(z)F(z))dA(z)\right)^{1+\epsilon},$$

with C and ϵ dependent only on the A_2 constant.

Proof. For each k define

$$E_k = \left\{z \in \mathbb{D} : M_\Delta(\operatorname{tr}(F^*F))(z) > 2^{4k+1}\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z)\right\}.$$

By Lemma 3.1.14 we can see that

$$\begin{aligned} M_\Delta \operatorname{tr}(F^*(0)F(0)) &\leq (4/3)^2 \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z))dA(z) \\ &< 2^{4k+1}\int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z))dA(z) \end{aligned}$$

for all k. So we know that each E_k is not the whole disk (as 0 is not contained in it) and hence we can do a Calderon-Zygmund decomposition.

So, for each E_k we have a disjoint union of dyadic rectangles Q_i whose union is equal to E_k and

$$\begin{aligned} 2^{4k+1}\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z) &< \frac{1}{|Q_i|}\int_{Q_i}\operatorname{tr}(F^*(z)F(z))dA(z) \\ &< 2^{4(k+1)}\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z). \end{aligned}$$

Two inequalities we will use from this are:

$$|Q_i| < 2^{-4k-1}\left\{\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z)\right\}^{-1}\int_{Q_i}\operatorname{tr}(F^*(z)F(z))dA(z)$$

and

$$\int_{Q_i}\operatorname{tr}(F^*(z)F(z))dA(z) < |Q_i|2^{4(k+1)}\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z).$$

We now take a maximal dyadic rectangle Q in E_{k-1} (which is larger than E_k) and note that

$$|E_k \cap Q| = \sum_{Q_i \subset Q}|Q_i|$$

(where the Q_i denote the maximal dyadic rectangles in E_k), which can be estimated by

$$\begin{aligned} &\sum_{Q_i\subset Q} 2^{-4k-1}\left\{\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z)\right\}^{-1}\int_{Q_i}\operatorname{tr}(F^*(z)F(z))dA(z) \\ &\leq 2^{-4k-1}\left\{\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z)\right\}^{-1}\int_{Q}\operatorname{tr}(F^*(z)F(z))dA(z) \end{aligned}$$

due to the dyadic decomposition of E_k. But as Q is also part of a Calderon-Zygmund decomposition, this time for E_{k-1}, we can also see that

$$\int_{Q}\operatorname{tr}(F^*(z)F(z))dA(z) < |Q|2^{4k}\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z).$$

Putting the last two inequalities together we see that

$$\begin{aligned} &|E_k \cap Q| \\ &< 2^{-4k-1}\left\{\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z)\right\}^{-1}|Q|2^{4k}\int_{\mathbb{D}}(\operatorname{tr}(F^*(z)F(z)))dA(z) \\ &= \frac{1}{2}|Q|. \end{aligned}$$

We are now in a position to use Lemma 3.1.10 as $\operatorname{tr}(F^*(z)F(z)))$ satisfies the scalar A_2 condition and $|E_k \cap Q| \leq \frac{1}{2}|Q|$. So with $\frac{1}{2}$ being our δ in 3.1.10, we can deduce that

$$\mu(E_k \cap Q) < \lambda\mu(Q),$$

for some $0 < \lambda < 1$ independent of k, with $d\mu(z) = \operatorname{tr}(F^*(z)F(z))dA(z)$. We can now sum over all maximal dyadic rectangles in E_{k-1} and see that

$$\mu(E_k) = \sum_Q \mu(E_k \cap Q) < \lambda \sum_Q \mu(Q) = \lambda\mu(E_{k-1}).$$

Let us take a moment here to note that λ depends only on our A_2 bound of $\operatorname{tr}(F^*(z)F(z))$, we can see this from Lemma 3.1.10, and that this A_2 bound is controlled by the matrix A_2 bound for F^*F and the dimension.

We have established that for each $k \geq 1$, $\mu(E_k) < \lambda\mu(E_{k-1})$ and so

$$\begin{aligned} \mu(E_k) < \lambda^k \mu(E_0) &= \lambda^k \int_{E_0} \operatorname{tr}(F^*(z)F(z))dA(z) \\ &\leq \lambda^k \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z))dA(z). \end{aligned}$$

Now let us move on and look at $\int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z))^{1+\epsilon}dA(z)$ for some $\epsilon > 0$. From Lemma 3.1.14 we know that $\operatorname{tr}(F^*F)(z) \leq M_\Delta \operatorname{tr}(F^*F)(z)$ on the disk. So,

$$\begin{aligned} &\int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z))^{1+\epsilon}dA(z) \leq \int_{\mathbb{D}} \operatorname{tr}(F^*(z)F(z)) \left\{M_\Delta \operatorname{tr}(F^*F)(z)\right\}^\epsilon dA(z) \\ &= \int_{x: M_\Delta \operatorname{tr}(F^*F)(x) \leq 2\int_{\mathbb{D}} \operatorname{tr}(F^*F(z))dA(z)} \operatorname{tr}(F^*(z)F(z)) \left\{M_\Delta \operatorname{tr}(F^*F)(z)\right\}^\epsilon dA(z) \\ &\qquad + \sum_k \int_{E_k - E_{k+1}} \operatorname{tr}(F^*(z)F(z)) \left\{M_\Delta \operatorname{tr}(F^*F)(z)\right\}^\epsilon dA(z) \end{aligned}$$

$$\begin{aligned} &\leq 2^\epsilon \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^{1+\epsilon} + \sum_k 2^{(4(k+1)+1)\epsilon} \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^\epsilon \mu(E_k) \\ &\leq 2^\epsilon \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^{1+\epsilon} \\ &+ \sum_k 2^{(4(k+1)+1)\epsilon} \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^\epsilon \lambda^k \int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z) \\ &= 2^\epsilon \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^{1+\epsilon} + \sum_k 2^{(4(k+1)+1)\epsilon} \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^{1+\epsilon} \lambda^k \\ &= \left\{\int_{\mathbb{D}} \operatorname{tr}(F^*F)(z)dA(z)\right\}^{1+\epsilon} \left(2^\epsilon + 2^{5\epsilon} \sum_k (\lambda 2^{4\epsilon})^k\right). \end{aligned}$$

If we choose ϵ such that $0 < \lambda 2^{4\epsilon} < 1$ then this will become

$$\left\{\int_{\mathbb{D}} \mathrm{tr}(F^*F)(z)dA(z)\right\}^{1+\epsilon} \left(2^{\epsilon} + 2^{5\epsilon}\frac{1}{1-\lambda 2^{4\epsilon}}\right),$$

thus for any $0 < \epsilon' \leq \epsilon$ our reverse Hölder inequality will hold.

□

Corollary 3.1.16. *If F^*F satisfies A_2 and J is a positive matrix, then there exists $\epsilon > 0$ such that*

$$\int (\mathrm{tr}(JF^*(z)F(z)J))^{1+\epsilon} dA(z) \leq C\left(\int \mathrm{tr}(JF^*(z)F(z)J)dA(z)\right)^{1+\epsilon}.$$

The same ϵ and constant C hold for all positive matrices J, and ϵ depends only on the dimension and the A_2 constant of F^*F.

Proof. This follows from 3.1.15 and 3.1.11. □

3.1.3 Proof of Theorem 3.1.1.

Two easy lemmas follow before the proof of the Theorem 3.1.1.

Lemma 3.1.17. *Let F and G be matrices consisting of Bergman space $L^2_a(\mathbb{D})$ functions. If $FG^*GF^* \geq \eta I$ and $T_F T_{G^*}$ is bounded, then the Toeplitz product $T_F T_{G^*}$ is invertible.*

Proof. $FG^*GF^* \geq \eta I$ implies that $G^{*-1}F^{-1}F^{*-1}G^{-1}$ is bounded and so the operator $T_{G^{*-1}F^{-1}} = T_{G^{*-1}}T_{F^{-1}}$ is bounded. It remains to note that

$$(T_F T_{G^*})T_{G^{*-1}}T_{F^{-1}}Fk_w e_1 = Fk_w e_1$$

and

$$T_{G^{*-1}}T_{F^{-1}}(T_F T_{G^*})k_w e_1 = k_w e_1,$$

and that these also hold for $k_w e_n$. The implication holds because the linear spans of $\{F(k_w e_n)\}$ and $\{(k_w e_n)\}$ form dense subspaces. □

Lemma 3.1.18. *If the trace of a positive matrix A is less than some constant $\lambda > 0$, then $A < CI$ for some constant $C > 0$ depending only on λ and the dimension, where I is the identity matrix.*

Proof. Trivial. □

Proof of Theorem 3.1.1. First we deal with the "$\Leftarrow$" direction of the proof:
From Lemmas 3.1.3 and 3.1.6 we know that F^*F satisfies our A_2 condition. Then by Corollary 3.1.16,

$$\int (\mathrm{tr}(((G^*G)(x))^{\frac{1}{2}}(F^*F)(z)((G^*G)(x))^{\frac{1}{2}}))^{1+\epsilon}dA(z) \leq \tag{3.1.1}$$
$$c\int (\mathrm{tr}(((G^*G)(x))^{\frac{1}{2}}(F^*F)(z)((G^*G)(x))^{\frac{1}{2}}))dA(z)^{1+\epsilon}$$

holds for all $x \in \mathbb{D}$ with some $\epsilon > 0$ and a constant c independent of x. Note here that we need to use the fact that G^*G is strictly positive.

If we note that for a positive matrix A, $AA^* \geq \eta > 0 \Rightarrow A^*A \geq \eta > 0$, then we can see that the hypothesis of Lemma 3.1.3 is symmetric in F and G. Then we can also apply Lemmas 3.1.3 and 3.1.6 to see that G^*G satisfies our A_2 condition. So a similar reverse Hölder will hold:

$$\int \left(\mathrm{tr}\left(\left\{\int(F^*F)(z)dA(z)\right\}^{\frac{1}{2}}(G^*G)(x)\left\{\int(F^*F)(z)dA(z)\right\}^{\frac{1}{2}}\right)\right)^{1+\epsilon'} dA(x)$$

$$\leq c\left(\int \mathrm{tr}\left(\left\{\int(F^*F)(z)dA(z)\right\}^{\frac{1}{2}}(G^*G)(x)\left\{\int(F^*F)(z)dA(z)\right\}^{\frac{1}{2}}\right)dA(x)\right)^{1+\epsilon'},$$

for some $\epsilon' > 0$. So let us set $\epsilon = \min\{\epsilon, \epsilon'\}$.

Thus, integrating both sides of the reverse Hölder inequality (3.1.1) with respect to x, we give

$$\begin{aligned}
&\int\int (\mathrm{tr}(((G^*G)(x))^{\frac{1}{2}}(F^*F)(z)((G^*G)(x))^{\frac{1}{2}}))^{1+\epsilon}dA(z)dA(x)\\
&\leq c\int\left\{\int(\mathrm{tr}(((G^*G)(x))^{\frac{1}{2}}(F^*F)(z)((G^*G)(x))^{\frac{1}{2}}))dA(z)\right\}^{1+\epsilon}dA(x)\\
&= c\int\left(\mathrm{tr}\left(((G^*G)(x))^{\frac{1}{2}}\int(F^*F)(z)dA(z)((G^*G)(x))^{\frac{1}{2}}\right)\right)^{1+\epsilon}dA(x)\\
&= c\int\left(\mathrm{tr}\left(\left(\int(F^*F)(z)dA(z)\right)^{\frac{1}{2}}(G^*G)(x)\left(\int(F^*F)(z)dA(z)\right)^{\frac{1}{2}}\right)\right)^{1+\epsilon}dA(x),
\end{aligned}$$

and so, as G^*G also has the A_2 condition, we can use our reverse Hölder again to see that this last expression is less than or equal to

$$c\left\{\int \mathrm{tr}\left(\left\{\int(F^*F)(z)dA(z)\right\}^{\frac{1}{2}}(G^*G)(x)\left\{\int(F^*F)(z)dA(z)\right\}^{\frac{1}{2}}\right)dA(x)\right\}^{1+\epsilon},$$

where, as usual, c is a constant that possibly changes from line to line. By the Möbius invariance of the Berezin transform, see [43] page 143, we see that

$$\int\int(\operatorname{tr}(((G^*G)(x))^{\frac{1}{2}}(F^*F)(z)((G^*G)(x))^{\frac{1}{2}}))^{1+\epsilon}|k_w(x)|^2|k_w(z)|^2dA(z)dA(x)$$
$$\leq c(\operatorname{tr}((B(G^*G)(w))^{\frac{1}{2}}B(F^*F)(w)(B(G^*G)(w))^{\frac{1}{2}}))^{1+\epsilon} < cM^{1+\epsilon}.$$

Hence, by Theorem 2.0.2, we can see that the Toeplitz product $T_FT_{G^*}$ is bounded. The invertibility of this Toeplitz product follows from Lemma 3.1.17.

Now we move on to the "$\Rightarrow$" direction of the proof.

If $T_FT_{G^*}$ is bounded and invertible, then we know from Theorem 2.0.3 that $\operatorname{tr}(B(F^*F)(w)B(G^*G)(w))$ is uniformly bounded and that $T_FT_{G^*}$ is bounded below. Thus in particular

$$\int \langle T_FT_{G^*}k_w\mathbf{e}, T_FT_{G^*}k_w\mathbf{e}\rangle\, dA(z) \geq \eta \int \langle k_w\mathbf{e}, k_w\mathbf{e}\rangle\, dA(z) = \eta\, \langle \mathbf{e}, \mathbf{e}\rangle$$

for all vectors $\mathbf{e} \in \mathbb{C}^n$. We know that $T_FT_{G^*}k_w = F(z)G^*(w)k_w(z)$ and so we deduce that $G(w)B(F^*F)(w)G^*(w) > \eta I$. From the fact that $||(T_FT_{G^*})^*||$ is also bounded below we can see that $F(w)B(G^*G)(w)F^*(w) > \eta I$. From these we deduce that

$$B(G^*G)(w) > \eta F^{-1}(w)F^{*-1}(w)$$

and

$$B(F^*F)(w) > \eta G^{-1}G^{*-1}(w),$$

which lets us see that

$$\left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}} B(G^*G)(w) \left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}}$$
$$> \eta \left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}} F^{-1}(w)F^{*-1}(w) \left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}},$$

and also

$$\{B(G^*G)(w)\}^{\frac{1}{2}} B(F^*F)(w) \{B(G^*G)(w)\}^{\frac{1}{2}}$$
$$> \eta \{B(G^*G)(w)\}^{\frac{1}{2}} G^{-1}(w)G^{*-1}(w) \{B(G^*G)(w)\}^{\frac{1}{2}},$$

thus

$$
\begin{aligned}
&\operatorname{tr}(B(G^*G)(w)B(F^*F)(w))\\
&= \operatorname{tr}(\{B(G^*G)(w)\}^{\frac{1}{2}}\, B(F^*F)(w)\,\{B(G^*G)(w)\}^{\frac{1}{2}})\\
&> \eta\operatorname{tr}(\{B(G^*G)(w)\}^{\frac{1}{2}}\, G^{-1}G^{*-1}\,\{B(G^*G)(w)\}^{\frac{1}{2}}))\\
&= \eta(\operatorname{tr}(\left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}}\, B(G^*G)(w)\,\left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}}))\\
&> \eta^2(\operatorname{tr}(\left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}}\, F^{-1}(w)F^{*-1}(w)\,\left\{G^{-1}G^{*-1}(w)\right\}^{\frac{1}{2}})).
\end{aligned}
$$

Thus, as $\operatorname{tr}(B(G^*G)(w)B(F^*F)(w))$ is uniformly bounded, $\operatorname{tr}(G^{*-1}(w)F^{-1}(w)F^{*-1}(w)G^{-1}(w))$ is uniformly bounded, by λ say, and so $G^{*-1}(w)F^{-1}(w)F^{*-1}(w)G^{-1}(w) < \lambda' I$, which gives us that $F(w)G^*(w)G(w)F^*(w) > \frac{1}{\lambda'}I$.

$\square$

Chapter 4

The Vector Martingale Transform, Band Operators and The Hilbert Transform: $L^2(\mathbb{C}^n, V) \to L^2(\mathbb{C}^n, U)$ Boundedness

4.1 Introduction

In this chapter we will give sufficient conditions for various dyadic operators to be bounded between $L^2(\mathbb{C}^n, V)$ and $L^2(\mathbb{C}^n, U)$. We will also give sufficient conditions for the Hilbert transform, and some other singular integral operators to be bounded here. The starting point for our discussion will be the matrix weight analogue of the A_2, and more general A_p conditions. Fundamental to our hypothesis will be a generalization of the A_∞ condition. There are direct analogues of the A_p conditions for matrix weights, but in order to define an A_∞ condition we need to look at the more general $A_{p,q}$ classes of weights. When $\frac{1}{p} + \frac{1}{q} = 1$ the $A_{p,q}$ class is a matrix generalization of the A_p class. We will, however, only concern ourselves with the $A_{2,q}$ classes of weights. For discussion of the more general classes see [21].

Definition 4.1.1. Let $q > 0$ and let U be a matrix weight. We say that U is in the class of matrix weights $A_{2,q}$ if

$$\sup_I || \left\langle U^{-\frac{q}{2}} \right\rangle_I^{\frac{1}{q}} \langle U \rangle_I^{\frac{1}{2}} || < C_q.$$

The class of matrix weights $A_{2,2}$ corresponds to the scalar A_2 class of weights and characterises those weights for which the Hilbert Transform is bounded $L^2(U) \to L^2(U)$, this was originally shown in [38].

Rather than there being one generalization of the A_∞ class of weights, there is a generalization for each p. In our case, $p = 2$, it is, in some sense, the limit of the $A_{2,q}$ class as $q \to 0$. For a formal treatment of the relationship between the $A_{p,q}$ classes and the $A_{p,0}$ classes again see [21]. Note the similarity of this condition to the matrix A_2 condition in the previous chapter.

Definition 4.1.2. We say that a matrix weight U is in the $A_{2,0}$ class of weights if there exists a constant $C > 0$ such that

$$\det \langle U \rangle_I \leq C \exp\{\langle \log \det U \rangle_I\}$$

for all intervals $I \in \mathbb{R}$. The reverse inequality always holds by Jensen's inequality for matrices, see page 48 of [21]. Similarly U is in the *dyadic* $A_{2,0}$ class of weights if the inequality holds for all dyadic intervals I.

Bownik [3] reformulates this property and introduces it in the context of $A_{q,p}$ matrix weights. Note the obvious generalization of the scalar (dyadic) A_∞ condition, again uniform over all (dyadic) intervals I:

$$\langle w \rangle_I \leq C \exp \langle \log w \rangle_I .$$

We also have the following weaker class of matrix weights.

Definition 4.1.3. We say that a matrix weight U is in the class $A^w_{2,0}$ of weights if the there exists $C > 0$ such that

$$\frac{1}{|I|} \int_I ||U^{\frac{1}{2}} x||^2 \leq C \exp \left\{ \left\langle \log ||U^{\frac{1}{2}} x||^2 \right\rangle_I \right\}$$

for all $x \in \mathbb{C}^n$ and intervals I. Again U is in the *dyadic* $A^w_{2,0}$ class of weights if the inequality holds for all dyadic intervals I. In simpler terms this $A^w_{2,0}$ class is the class of matrix weights U such that $\langle Ux, x \rangle$ is a scalar A_∞ weight for each $x \in \mathbb{C}^n$, $x \neq 0$, with uniformly bounded A_∞ constant.

Definition 4.1.4. The inverse volume, v^{-1}, of a positive homogeneous function $\theta : \mathbb{C}^n \to \mathbb{R}_+$ is defined as

$$v^{-1}(\theta) = \frac{\text{vol}(\{x \in \mathbb{C}^n : ||x|| \leq 1\})}{\text{vol}(\{x \in \mathbb{C}^n : \theta(x) \leq 1\})}.$$

If A is a positive matrix, then $v^{-1}(||A\cdot||) = (\det(A))^2$. It then follows that

$$v^{-1}\left(||\langle U\rangle_I^{\frac{1}{2}}\cdot||^2\right) = v^{-1}\left(||\langle U\rangle_I^{\frac{1}{2}}\cdot||\right) = \left(\det\langle U\rangle_I^{\frac{1}{2}}\right)^2 = (\det\langle U\rangle_I).$$

This allows us to reformulate the matrix $A_{2,0}$ class as being the class of matrix weights U for which there exists a constant $C > 0$ such that

$$v^{-1}\left(||\langle U\rangle_I^{\frac{1}{2}}\cdot||^2\right) \le C\exp\{\langle\log\det U\rangle_I\}$$

for all intervals I. Likewise, the dyadic $A_{2,0}$ condition is reformulated but with the inequality restricted to dyadic intervals I. This expresses the $A_{2,0}$ condition, taking $V = U^{\frac{1}{2}}$, in the form of the hypothesis of Lemma 3.2 in [3], and thus the following lemma follows immediately from Lemmas 3.2 and 3.3 in [3].

Lemma 4.1.5. $A_{2,0} \subset A_{2,0}^w$.

4.2 The $A_{2,0}$ Condition and Reverse Hölder

Definition 4.2.1. A matrix weight U satisfies the *reverse Hölder inequality* if there exists a constant $C > 0$ and $r > 2$ such that

$$\int_I ||U^{\frac{1}{2}}(x)\langle U\rangle_I^{-\frac{1}{2}}y||^r dx \le C|I|||y||^r$$

holds for all intervals I and nonzero vectors $x \in \mathbb{C}^n$. As with previous definitions we have a weaker *dyadic reverse Hölder inequality* where the inequality holds for all dyadic intervals.

The opposite inequality, with constant 1, follows from Hölder's inequality. Note that our definition of the reverse Hölder property is weaker, if stated in terms of bounded operators on a Hilbert space, than the existing definition in the literature [4]. It is, however, equivalent in the case of finite dimensional spaces. Our definition generalizes the scalar reverse Hölder condition, see page 201 of [32].

Lemma 4.2.2. *A matrix weight U satisfies the $A_{2,0}^w$ condition if and only if it satisfies the reverse Hölder inequality.*

Proof. We have that

$$\frac{1}{|I|}\int_I ||U^{\frac{1}{2}}x||^2 \le C\exp\left\{\left\langle\log||U^{\frac{1}{2}}x||^2\right\rangle_I\right\} \tag{4.2.1}$$

for all nonzero x and intervals I, and thus the scalar weight $||U^{\frac{1}{2}}x||^2$ satisfies the A_∞ condition and hence a reverse Hölder inequality:

$$\left\{\frac{1}{|I|}\int_I ||U^{\frac{1}{2}}x||^{2r}\right\}^{\frac{1}{2r}} \leq C\left\{\frac{1}{|I|}\int_I ||U^{\frac{1}{2}}x||^2\right\}^{\frac{1}{2}}$$

for some $r > 1$, all intervals I and all nonzero x. Note that the index r does not depend on x because it only depends on the A_∞ constant C in (4.2.1), which is uniform for all x. As this is true for all nonzero x we can replace x by $\langle U\rangle_I^{-\frac{1}{2}} y$, where $0 \neq y \in \mathbb{C}^n$. Thus for all intervals $I \in \mathbb{R}$ and $y \in \mathbb{C}^n$

$$\left\{\frac{1}{|I|}\int_I ||U^{\frac{1}{2}}\langle U\rangle_I^{-\frac{1}{2}} y||^{2r}\right\}^{\frac{1}{2r}} \leq C\left\{\frac{1}{|I|}\int_I ||U^{\frac{1}{2}}\langle U\rangle_I^{-\frac{1}{2}} y||^2\right\}^{\frac{1}{2}} = C||y||.$$

Note that each of the above steps concerns equivalent statements, and so the reverse implication holds. □

Lemma 4.2.3. *A matrix weight U satisfies the dyadic $A_{2,0}^w$ condition if and only if it satisfies the dyadic reverse Hölder inequality.*

Proof. The proof is identical to that of the previous lemma. □

We present one final definition before moving on to our main theorems.

Definition 4.2.4. We say that for matrix weights U and V (U,V) is an A_2 pair if for some constant multiple of the identity, $C > 0$,

$$\left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} \langle U\rangle_I \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} < C$$

for all dyadic intervals I.

4.3 Boundedness of the martingale transform

We are now able to state our main theorem concerning sufficient conditions for the boundedness of the dyadic martingale transforms.

Theorem 4.3.1. *Let (U,V) be an A_2 pair of matrix weights. If $V^{-1} \in A_{2,0}$ and U satisfies the dyadic $A_{2,0}^w$ condition, then the dyadic martingale transforms are uniformly bounded from $L^2(V)$ to $L^2(U)$.*

Corollary 4.3.2. *Let (U,V) be an A_2 pair of matrix weights. If U and V^{-1} are both in $A_{2,0}$, then the dyadic martingale transforms are uniformly bounded from $L^2(V)$ to $L^2(U)$.*

Proof. By Lemma 4.1.5, Theorem 4.3.1 implies this corollary.

□

The conditions on the matrix weights U and V^{-1} are symmetric in this corollary. It is worth noting that in Theorem 6.1 of [17] the conditions and implications in Corollary 4.3.2 and Theorem 4.3.1 are stated but specifically for the scalar-valued function space setting, this is also mentioned in [22].

4.4 Proof of Theorem 4.3.1 using a Two-Weighted Dyadic Square Function

For a matrix weight V, we define the operator $D_{V^{-1}}$ by

$$D_{V^{-1}}f = D_{V^{-1}}\sum_{I\in\mathcal{D}} f_I h_I(x) \mapsto \sum_{I\in\mathcal{D}} \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} f_I h_I(x)$$

for $\mathbb{C}^n$-valued functions f with finite Haar expansion.

If we write $M_V^{-\frac{1}{2}} T_\sigma M_U^{\frac{1}{2}}$ as $M_V^{-\frac{1}{2}} T_\sigma D_{V^{-1}}^{-1} D_{V^{-1}} M_U^{\frac{1}{2}}$ and note that T_σ and $D_{V^{-1}}^{-1}$ commute, then this allows us to estimate the norm as

$$||M_V^{-\frac{1}{2}} T_\sigma M_U^{\frac{1}{2}}|| = ||M_V^{-\frac{1}{2}} T_\sigma D_{V^{-1}}^{-1} D_{V^{-1}} M_U^{\frac{1}{2}}|| \leq ||M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}|| ||T_\sigma|| ||D_{V^{-1}} M_U^{\frac{1}{2}}||.$$

We know that T_σ is bounded on unweighted $L^2(\mathbb{R},\mathbb{C}^n)$ so we seek conditions on the matrix weights U and V^{-1} that imply the boundedness of the operators $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$ and $D_{V^{-1}} M_U^{\frac{1}{2}}$ on unweighted $L^2(\mathbb{R},\mathbb{C}^n)$.

We deal with $D_{V^{-1}} M_U^{\frac{1}{2}}$, a two-weighted dyadic square function, using a stopping time argument and Cotlar's Lemma.

Here is our first main result.

Theorem 4.4.1. *Let U and V^{-1} be matrix weights such that U has the* dyadic reverse Hölder inequality *and such that for all dyadic intervals I,*

$$\left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} \langle U\rangle_I \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} < C.$$

Then the two-weighted square function $S = M_U^{\frac{1}{2}} D_{V^{-1}}$ is bounded on $L^2(\mathbb{R},\mathbb{C}^n)$.

There is a proof of the next theorem by Fedor Nazarov and Sergei Treil in [21]. Their proof uses a Bellman function technique, but we will present a proof without using this technique.

Theorem 4.4.2. *Let V^{-1} be a matrix weight such that $V^{-1} \in A_{2,0}$. Then $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$ is bounded on $L^2(\mathbb{R}, \mathbb{C}^n)$.*

We now introduce the stopping time used in the proof of Theorem 4.4.1.

4.4.1 Stopping Time

Let $\lambda > 1$ and let $\mathcal{J}_{\lambda,1}(J)$ be the collection of maximal dyadic subintervals I_λ of J such that

$$||\frac{1}{|I_\lambda|}\int_{I_\lambda} \langle V^{-1}\rangle_J^{\frac{1}{2}} U(x) \langle V^{-1}\rangle_J^{\frac{1}{2}} dx|| > \lambda \tag{4.4.1}$$

or

$$||\frac{1}{|I_\lambda|}\int_{I_\lambda} \langle V^{-1}\rangle_J^{-\frac{1}{2}} V^{-1}(x) \langle V^{-1}\rangle_J^{-\frac{1}{2}} dx|| > \lambda \tag{4.4.2}$$

or

$$||\frac{1}{|I_\lambda|}\int_{I_\lambda} \langle U\rangle_J^{-\frac{1}{2}} U(x) \langle U\rangle_J^{-\frac{1}{2}} dx|| > \lambda. \tag{4.4.3}$$

Then we define inductively $\mathcal{J}_{\lambda,k}(J)$ as $\cup_{I\in\mathcal{J}_{\lambda,k-1}(J)}\mathcal{J}_{\lambda,1}(I)$ for $k > 1$. Let $\mathcal{F}_{\lambda,1}(J)$ be the collection of those dyadic subintervals of J which are not a subinterval of any interval in $\mathcal{J}_{\lambda,1}(J)$. We likewise define $\mathcal{F}_{\lambda,k}(J)$ iteratively to be $\cup_{I\in\mathcal{J}_{\lambda,k-1}(J)}\mathcal{F}_{\lambda,1}(I)$. Then $\cup_k\mathcal{F}_{\lambda,k}(J)$ forms a decomposition of the dyadic subintervals of J.

Lemma 4.4.3. *If U and V are matrix weights such that (U, V) is an A_2 pair , then $\mathcal{J}$ is a decaying stopping time for some $\lambda > 1$. By decaying stopping time we mean that we have a constant $0 < \delta < 1$ such that $|\mathcal{J}(I)_{\lambda,k}| \leq \delta^k|I|$ for all k.*

Proof. We first restrict ourselves to showing that $\mathcal{J}_{\lambda,k}(J) = \cup_{I\in\mathcal{J}_{\lambda,k-1}(J)}\mathcal{J}_{\lambda,1}(I)$ is a decaying stopping time when $\mathcal{J}_{\lambda,1}(I)$ is defined as the collection of maximal subintervals of I satisfying only (4.4.3) rather than all three conditions.

We have the following series of inequalities:

$$|I| \geq ||\int_I \langle U\rangle_I^{-\frac{1}{2}} U(x) \langle U\rangle_I^{-\frac{1}{2}} dx|| \geq ||\sum_{J\in\mathcal{J}_{\lambda,1}} \int_J \langle U\rangle_I^{-\frac{1}{2}} U(x) \langle U\rangle_I^{-\frac{1}{2}} dx||$$

$$\geq C_n \sum_{J\in\mathcal{J}_{\lambda,1}} ||\int_J \langle U\rangle_I^{-\frac{1}{2}} U(x) \langle U\rangle_I^{-\frac{1}{2}} ||,$$

where C_n is a constant dependent on the dimension. This holds due to the equivalence of all matrix norms and the additivity of the trace norm on positive matrices. By (4.4.3),

$$C_n \sum_{J\in\mathcal{J}_{\lambda,1}} ||\int_J \langle U\rangle_I^{-\frac{1}{2}} U(x) \langle U\rangle_I^{-\frac{1}{2}} || \geq C_n\lambda \sum_{J\in\mathcal{J}_{\lambda,1}} |J|,$$

and hence

$$\frac{1}{\lambda C_n}|I| \geq \sum_{J\in\mathcal{J}_{\lambda,1}} |J| = |\mathcal{J}_{\lambda,1}|.$$

Thus we can choose λ to be large enough such that $\frac{1}{\lambda C_n} < 1$ and we have $|\mathcal{J}_{\lambda,1}(I)| < \delta|I|$. Iteration now yields that $|\mathcal{J}_{\lambda,k}(I)| < \delta^k|I|$. We use a similar argument for 4.4.1 and 4.4.2 individually and then note that the finite union of decaying stopping times will also be a decaying stopping time, after a possible change of λ. This is possible because we can alter the λ to make δ as small as we wish. So for each of the three individual stopping times we make sure that the corresponding δ is strictly less than $\frac{1}{3}$. The union of these stopping times will then be a stopping time with $\delta < 1$.

□

4.4.2 Proof of Theorem 4.4.1

The proof of this theorem is where the core of our technical analysis takes place, it draws from Theorem 3.1 in [29]. We are presenting a generalization for the finite dimensional case. We make use of a version of Cotlar's lemma.

Lemma 4.4.4. *[Cotlar] Suppose we have an operator $T = \sum_i T_i$ on a Hilbert space $\mathcal{H}$ and A such that*

- $||T_i|| \leq A$
- $||T_i^* T_j|| \leq \theta(i-j)$, *if* $i \neq j$
- $||T_i T_j^*|| = 0$, *if* $i \neq j$

where θ is a function such that $\sum_i \theta(i) = A$, then $||T|| \leq \sqrt{2}A$.

Proof. See page 195 of [7]. □

Proof of Theorem 4.4.1. We choose $\lambda > 0$ such that the stopping time in 4.4.3, $\mathcal{J}$, is a decaying stopping time. First note that almost everywhere on $J \backslash \cup \mathcal{J}(J)$

$$\left\langle V^{-1}\right\rangle_J^{\frac{1}{2}} U(x) \left\langle V^{-1}\right\rangle_J^{\frac{1}{2}} \leq \lambda,$$

$$\left\langle U\right\rangle_J^{-\frac{1}{2}} U(x) \left\langle U\right\rangle_J^{-\frac{1}{2}} \leq \lambda$$

and

$$\left\langle V^{-1}\right\rangle_J^{-\frac{1}{2}} V^{-1}(x) \left\langle V^{-1}\right\rangle_J^{-\frac{1}{2}} \leq \lambda.$$

In this context λ stands for the identity matrix scaled by λ, and the inequalities are matrix inequalities. Let us take $f \in L^2(\mathbb{R}, \mathbb{C}^n)$ with finite Haar expansion. Assume without loss that f is supported in the unit interval. We write $\mathcal{J}_j$ and $\mathcal{F}_j$ for $\mathcal{J}_{\lambda,j}([0,1])$ and $\mathcal{F}_{\lambda,j}([0,1])$. Define

$$\triangle_j f = \sum_{K \in \mathcal{F}_j} h_K f_K$$

and

$$S_j f = S\triangle_j f = U^{\frac{1}{2}} \sum_{K \in \mathcal{F}_j} \left\langle V^{-1} \right\rangle_K^{\frac{1}{2}} h_K f_K.$$

We can check that $\sum_{j=1}^{\infty} \triangle_j f = f$ and also that

$$\sum_{j=1}^{\infty} S_j f = Sf.$$

We show that S is bounded using Cotlar's Lemma. First note that

$$||S_j f||_{L^2}^2 = \int_{\cup \mathcal{J}_{j-1}} ||S_j f||_{\mathbb{C}^n}^2 dx = \int_{\cup \mathcal{J}_{j-1} \setminus \cup \mathcal{J}_j} ||S_j f||_{\mathbb{C}^n}^2 dx + \int_{\cup \mathcal{J}_j} ||S_j f||_{\mathbb{C}^n}^2 dx.$$

We estimate $\int_{\cup \mathcal{J}_{j-1} \setminus \cup \mathcal{J}_j} ||S_j f||_{\mathbb{C}^n}^2 dx$ and then $\int_{\cup \mathcal{J}_j} ||S_j f||_{\mathbb{C}^n}^2 dx$.

$$\begin{aligned}
\int_{\cup \mathcal{J}_{j-1} \setminus \cup \mathcal{J}_j} ||S_j f||_{\mathbb{C}^n}^2 dx &= \sum_{J \in \mathcal{J}_{j-1}} \int_{J \setminus \cup \mathcal{J}(J)} ||S_j f||_{\mathbb{C}^n}^2 dx \\
&= \sum_{J \in \mathcal{J}_{j-1}} \int_{J \setminus \cup \mathcal{J}(J)} ||U^{\frac{1}{2}}(x) \sum_{K \in \mathcal{F}(J)} \left\langle V^{-1} \right\rangle_K^{\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx \\
&= \sum_{J \in \mathcal{J}_{j-1}} \int_{J \setminus \cup \mathcal{J}(J)} ||U^{\frac{1}{2}}(x) \left\langle V^{-1} \right\rangle_J^{\frac{1}{2}} \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \sum_{K \in \mathcal{F}(J)} \left\langle V^{-1} \right\rangle_K^{\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx \\
&\leq \sum_{J \in \mathcal{J}_{j-1}} \int_{J \setminus \cup \mathcal{J}(J)} ||U^{\frac{1}{2}}(x) \left\langle V^{-1} \right\rangle_J^{\frac{1}{2}} ||^2 || \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \sum_{K \in \mathcal{F}(J)} \left\langle V^{-1} \right\rangle_K^{\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx \\
&\leq \sum_{J \in \mathcal{J}_{j-1}} \int_{J \setminus \cup \mathcal{J}(J)} \lambda || \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \sum_{K \in \mathcal{F}(J)} \left\langle V^{-1} \right\rangle_K^{\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx \\
&\leq \sum_{J \in \mathcal{J}_{j-1}} \int_J \lambda \sum_{K \in \mathcal{F}(J)} || \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \left\langle V^{-1} \right\rangle_K^{\frac{1}{2}} ||^2 ||f_K||_{\mathbb{C}^n}^2 dx \\
&\leq \sum_{J \in \mathcal{J}_{j-1}} \int_J \lambda \sum_{K \in \mathcal{F}(J)} \lambda ||f_K||_{\mathbb{C}^n}^2 dx
\end{aligned}$$

since for $K \in \mathcal{F}(J)$

$$\left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \left\langle V^{-1} \right\rangle_K \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} = \frac{1}{|K|} \int_K \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} V^{-1}(x) \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \leq \lambda.$$

Thus

$$\int_{\cup\mathcal{J}_{j-1}\setminus\cup\mathcal{J}_j} ||S_j f||^2_{\mathbb{C}^n} dx \leq \sum_{J\in\mathcal{J}_{j-1}} \int_J \lambda \sum_{K\in\mathcal{F}(J)} \lambda ||f_K||^2_{\mathbb{C}^n} dx$$
$$= \sum_{J\in\mathcal{J}_{j-1}} \lambda^2 \sum_{K\in\mathcal{F}(J)} \int_J ||f_K||^2_{\mathbb{C}^n} dx = \lambda^2 ||\triangle_j f||_{L^2}.$$

We now consider

$$\int_{\cup\mathcal{J}_j} ||S_j f||^2_{\mathbb{C}^n} dx = \sum_{J\in\mathcal{J}_{j-1}} \sum_{I\in\mathcal{J}(J)} \int_I ||U^{\frac{1}{2}}(x) \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle^{\frac{1}{2}} f_K h_K(x)||^2_{\mathbb{C}^n} dx.$$

As h_K is constant on $I \in \mathcal{J}(J)$ for each $K \in \mathcal{F}(J)$, this is equal to

$$\sum_{J\in\mathcal{J}_{j-1}} \sum_{I\in\mathcal{J}(J)} \int_I \left\langle U^{\frac{1}{2}}(x) \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K, U^{\frac{1}{2}}(x) \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K \right\rangle_{\mathbb{C}^n} dx$$

$$= \sum_{J\in\mathcal{J}_{j-1}} \sum_{I\in\mathcal{J}(J)} \left\langle \int_I U(x) dx \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K, \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K \right\rangle_{\mathbb{C}^n}$$

$$= \sum_{J\in\mathcal{J}_{j-1}} \sum_{I\in\mathcal{J}(J)} |I| \left\langle \langle U\rangle_I \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K, \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K \right\rangle_{\mathbb{C}^n}$$

$$= \sum_{J\in\mathcal{J}_{j-1}} \sum_{I\in\mathcal{J}(J)} \int_I || \langle U\rangle_I^{\frac{1}{2}} \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K||^2_{\mathbb{C}^n} dx$$

$$\leq \sum_{J\in\mathcal{J}_{j-1}} \sum_{I\in\mathcal{J}(J)} \int_I || \langle U\rangle_I^{\frac{1}{2}} \langle U\rangle_J^{-\frac{1}{2}} ||^2_{\mathbb{C}^n} || \langle U\rangle_J^{\frac{1}{2}} \sum_{K\in\mathcal{F}(J)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} f_K h_K||^2_{\mathbb{C}^n} dx \leq 2\lambda^2 ||\triangle_j f||^2_{L^2}.$$

We have shown that there is a constant C such that $||S_j f||^2 \leq C||\triangle_j f||^2$. Let us now show that there exists a constant C' and $0 < d < 1$ such that for $k > j$

$$\int_{\cup\mathcal{J}_{k-1}} ||S_j f||^2 dx \leq C' d^{k-j} ||\triangle_j f||^2.$$

Cotlar's Lemma, Lemma 4.4.4, then implies that $S = \sum S_j$ is bounded. Note that

$$\int_{\cup\mathcal{J}_{k-1}} ||S_j f||^2 dx = \sum_{J\in\mathcal{J}_j} \sum_{I\in\mathcal{J}_{k-j-1}(J)} \int_I ||U^{\frac{1}{2}}(x) \sum_{L\in\mathcal{J}_{j-1}} \sum_{K\in\mathcal{F}(L)} \left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} h_K(x) f_K||^2_{\mathbb{C}^n} dx.$$

Note that $\sum_{L\in\mathcal{J}_{j-1}}\sum_{K\in\mathcal{F}(L)}\left\langle V^{-1}\right\rangle_K^{\frac{1}{2}} h_K f_K$ is constant on $J\in\mathcal{J}_j$, and denote this constant by $M_J f$. The above expression is equal to

$$\begin{aligned}
&\sum_{J\in\mathcal{J}_j}\sum_{I\in\mathcal{J}_{k-j-1}(J)}|I|||\langle U\rangle_I^{\frac{1}{2}} M_J f||^2_{\mathbb{C}^n}\\
&=\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\sum_{I\in\mathcal{J}_{k-j-1}(J)}|I|||\langle U\rangle_I^{\frac{1}{2}} M_J f||^2_{\mathbb{C}^n}\\
&=\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\sum_{I\in\mathcal{J}_{k-j-1}(J)}\left\langle |I|^{\frac{1}{2}}\langle U\rangle_I^{\frac{1}{2}} M_J f, |I|^{\frac{1}{2}}\langle U\rangle_I^{\frac{1}{2}} M_J f\right\rangle\\
&=\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\sum_{I\in\mathcal{J}_{k-j-1}(J)}\left\langle |I|^{\frac{1}{2}}\langle U\rangle_I^{\frac{1}{2}}\langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f, |I|^{\frac{1}{2}}\langle U\rangle_I^{\frac{1}{2}}\langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f\right\rangle\\
&=\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\left\langle\sum_{I\in\mathcal{J}_{k-j-1}(J)}|I|\langle U\rangle_I\langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f, \langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f\right\rangle\\
&=\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\int_{\mathcal{J}_{k-j-1}(J)}||U^{\frac{1}{2}}(x)\langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f||^2 dx.
\end{aligned}$$

We now apply Hölder's inequality with p such that $2p$ is the r from our reverse Hölder inequality on U. Then the above expression is less than or equal to

$$\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\left(\int_{\mathcal{J}_{k-j-1}(J)}||U^{\frac{1}{2}}(x)\langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f||^{2p}dx\right)^{\frac{1}{p}}|\mathcal{J}_{k-j-1}(J)|^{\frac{1}{q}}.$$

We now use the fact that we are working with a decaying stopping time to see that this is less than or equal to

$$\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}\left(\int_{\mathcal{J}_{k-j-1}(J)}||U^{\frac{1}{2}}(x)\langle U\rangle_J^{-\frac{1}{2}}\langle U\rangle_J^{\frac{1}{2}} M_J f||^{2p}dx\right)^{\frac{1}{p}} d^{\frac{k-j-1}{q}}|J|^{\frac{1}{q}}$$

where $0<d<1$. Now we apply the reverse Hölder inequality 4.2.1, with vector $\langle U\rangle_J^{\frac{1}{2}} M_J f$, to obtain that this is less than or equal to

$$\begin{aligned}
\sum_{J'\in\mathcal{J}_{j-1}}\sum_{J\in\mathcal{J}(J')}||\langle U\rangle_J^{\frac{1}{2}} M_J f||^2 C^{\frac{1}{p}}|J|^{\frac{1}{p}} d^{\frac{k-j-1}{q}}|J|^{\frac{1}{q}} &= d^{\frac{k-j-1}{q}} C^{\frac{1}{p}}\sum_{J'\in\mathcal{J}_{j-1}}\int_{\cup\mathcal{J}(J')}||U(x)^{\frac{1}{2}} M_J f||^2\\
&= d^{\frac{k-j-1}{q}} C^{\frac{1}{p}}\sum_{J'\in\mathcal{J}_{j-1}}\int_{\cup\mathcal{J}(J')}||S_j f||^2.
\end{aligned}$$

This is our core estimate.

To apply Cotlar's Lemma consider

$$\langle S_k^* S_j f, g\rangle_{L^2} = \langle S_j f, S_k g\rangle_{L^2}$$

$$= \int_{\cup \mathcal{J}_{k-1}} \langle S_j f(x), S_k g(x) \rangle_{\mathbb{C}^n} \, dx \leq \int_{\cup \mathcal{J}_{k-1}} ||S_j f(x)||_{\mathbb{C}^n} ||S_k g(x)||_{\mathbb{C}^n} dx$$

$$\leq \left\{ \int_{\cup \mathcal{J}_{k-1}} ||S_j f(x)||^2_{\mathbb{C}^n} \right\}^{\frac{1}{2}} \left\{ \int_{\cup \mathcal{J}_{k-1}} ||S_k g(x)||^2_{\mathbb{C}^n} dx \right\}^{\frac{1}{2}}$$

$$\leq d^{\frac{k-j-1}{2q}} C^{\frac{1}{2p}} \left\{ \int_{\cup \mathcal{J}_{j-1}} ||S_j f(x)||^2 dx \right\}^{\frac{1}{2}} \left\{ \int_{\cup \mathcal{J}_{k-1}} ||S_k g(x)||^2_{\mathbb{C}^n} dx \right\}^{\frac{1}{2}}$$

$$\leq d^{\frac{k-j-1}{2q}} C^{\frac{1}{2p}} ||\triangle_j f|| ||\triangle_k f|| \leq d^{\frac{k-j-1}{2q}} C^{\frac{1}{2p}} ||f||^2.$$

Note here that we are in the situation where $k > j$. If $j > k$, then we can look at the adjoint and apply the same inequality with the roles of j and k interchanged. Thus the function θ used in the application of Cotlar's lemma is $\theta(i-j) = d^{\frac{|k-j|-1}{2q}}$.

This is true as the support of $S_k f$ is contained in $\mathcal{J}_{k-1}$, and an application of Cauchy-Schwartz. Also note that

$$\left\langle S_k S_j^* f, g \right\rangle_{L^2} = \left\langle S_j^* f, S_k^* g \right\rangle_{L^2} = \langle (S\triangle_j)^* f, (S\triangle_k)^* g \rangle_{L^2} = \langle \triangle_j S^* f, \triangle_k S^* g \rangle_{L^2} = 0$$

when $k \neq j$ as the $\triangle_j$ are self-adjoint orthogonal projections. This finishes the proof of Theorem 4.4.1. $\square$

4.4.3 Stopping Time Part Two

We now introduce a stopping time, which we will show is decaying, which will be used in the proof of Theorem 4.4.2. Let $\lambda_2 > \lambda_1 > 1$ and let $\mathcal{I}_{\lambda,1}(J)$ be the maximal collection of subintervals of J such that for $I \in \mathcal{I}_{\lambda,1}(J)$, $|| \langle W \rangle_J^{-\frac{1}{2}} \langle W \rangle_I \langle W \rangle_J^{-\frac{1}{2}} || \leq \lambda_1$ and $\langle W \rangle_J^{-\frac{1}{2}} \langle W \rangle_I \langle W \rangle_J^{-\frac{1}{2}}$ has an eigenvalue less than $\frac{1}{\lambda_2}$. With the same $\lambda_2 > \lambda_1 > 1$, now let $\mathcal{I}'_{\lambda,1}(J)$ be the maximal collection of subintervals of J such that for $I \in \mathcal{I}'_{\lambda,1}(J)$, $|| \langle W \rangle_J^{-\frac{1}{2}} \langle W \rangle_I \langle W \rangle_J^{-\frac{1}{2}} || > \lambda_1$ and I is not a subinterval of any interval in $\mathcal{I}_{\lambda,1}(J)$. As before we define inductively $\mathcal{I}_{\lambda,k}(J) = \cup_{I \in \mathcal{I}_{\lambda,k-1}(J) \cup \mathcal{I}'_{\lambda,k-1}(J)} \mathcal{I}_{\lambda,1}(I)$ and similarly for $\mathcal{I}'_{\lambda,k}(J)$

Theorem 4.4.5. *If $W \in A_{2,0}$ then the union of $\mathcal{I}$ and $\mathcal{I}'$ is a decaying stopping time.*

Proof. We first show how we can choose λ_1 such that we have control over $|\mathcal{I}'_{\lambda,1}(J)|$.

$$|J| = || \langle W \rangle_J^{-\frac{1}{2}} \int_J W \langle W \rangle_J^{-\frac{1}{2}} ||$$

$$\geq || \langle W \rangle_J^{-\frac{1}{2}} \sum_{I \in \mathcal{I}'_{\lambda,1}(J)} \int_I W \langle W \rangle_J^{-\frac{1}{2}} || \geq \sum_{I \in \mathcal{I}'_{\lambda,1}(J)} C_n || \langle W \rangle_J^{-\frac{1}{2}} \int_I W \langle W \rangle_J^{-\frac{1}{2}} ||$$

$$= \sum_{I\in\mathcal{I}'_{\lambda,1}} (J) C_n || \langle W\rangle_J^{-\frac{1}{2}} \frac{|I|}{|I|} \int_I W \langle W\rangle_J^{-\frac{1}{2}} || \geq \sum_{I\in\mathcal{I}'_{\lambda,1}(J)} C_n |I| \lambda_1 = C_n \lambda_1 |\mathcal{I}'_{\lambda,1}(J)|.$$

We can now choose λ_1 to be large enough such that $\frac{\delta}{2}|J| > |\mathcal{I}'_{\lambda,1}(J)|$ for some fixed $\delta < 1$.

We now show that we can choose λ_1 and λ_2 such that we have similar control over $|\mathcal{I}_{\lambda,1}(J)|$. We will use $\mathcal{I}(J)$ to denote $\mathcal{I}_{\lambda,1}(J)$ here. Let $\mathcal{K}(J)$ denote the dyadic intervals in $J \setminus \{\mathcal{I}'(J) \cup \mathcal{I}(J)\}$ which are maximal. By Jensen's inequality for matrix functions we have the following three inequalities:

for $I \in \mathcal{I}'(J)$,

$$\left\langle \log\det\left\{\langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\}\right\rangle_I \leq \log\det\left\{\langle W\rangle_J^{-\frac{1}{2}} \langle W\rangle_I \langle W\rangle_J^{-\frac{1}{2}}\right\} \leq n\log(2\lambda_1); \quad (4.4.4)$$

for $I \in \mathcal{I}(J)$,

$$\left\langle \log\det\left\{\langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\}\right\rangle_I \leq \log\det\left\{\langle W\rangle_J^{-\frac{1}{2}} \langle W\rangle_I \langle W\rangle_J^{-\frac{1}{2}}\right\} \leq (n-1)\log(\lambda_1) - \log(\lambda_2); \quad (4.4.5)$$

for $I \in \mathcal{K}(J)$,

$$\left\langle \log\det\left\{\langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\}\right\rangle_I \leq \log\det\left\{\langle W\rangle_J^{-\frac{1}{2}} \langle W\rangle_I \langle W\rangle_J^{-\frac{1}{2}}\right\} \leq n\log(\lambda_1). \quad (4.4.6)$$

Note that the eigenvalues of the matrices in (4.4.6) and (4.4.4) are at most λ_1 and $2\lambda_1$, respectively. The inequality 4.4.4 follows from the maximality of I. All three inequalities depend on the equivalence of matrix norms and the use either the trace or the largest eigenvalue as a norm.

Using these inequalities and the $A_{2,0}$ condition for W, which implies the $A_{2,0}$ for $\langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}$ with the same constant, we can deduce the following series of inequalities:

$$1 = \det \langle W\rangle_J^{-\frac{1}{2}} \langle W\rangle_J \langle W\rangle_J^{-\frac{1}{2}} \leq C\exp\left\{\left\langle \log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\rangle_J\right\}$$

$$= C\exp\left\{\frac{1}{|J|}\left(\int_{\mathcal{I}(J)}\left(\log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right) + \int_{\mathcal{I}'(J)}\left(\log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right) + \int_{\mathcal{K}(J)}\left(\log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right)\right)\right\}$$

$$= C\exp\left\{\sum_{I\in\mathcal{I}(J)}\frac{|I|}{|J|}\left\langle \log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\rangle_I + \sum_{I\in\mathcal{I}'(J)}\frac{|I|}{|J|}\left\langle \log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\rangle_I + \right.$$
$$\left.\sum_{I\in\mathcal{K}(J)}\frac{|I|}{|J|}\left\langle \log\det \langle W\rangle_J^{-\frac{1}{2}} W \langle W\rangle_J^{-\frac{1}{2}}\right\rangle_I\right\}.$$

We now use the inequalities, (4.4.4),(4.4.5) and (4.4.6), to see that this can be estimated by:

$$C\exp\left\{\sum_{I\in\mathcal{I}'(J)}\frac{|I|}{|J|}n\log(2\lambda_1) + \sum_{I\in\mathcal{I}(J)}\frac{|I|}{|J|}((n-1)\log(\lambda_1)-\log(\lambda_2)) + \right.$$
$$\left.\sum_{I\in\mathcal{K}(J)}\frac{|I|}{|J|}n\log(\lambda_1)\right\}$$
$$\leq C\exp\left\{n\log(2\lambda_1) - \frac{|\mathcal{I}(J)|}{|J|}\log(\lambda_2)\right\} = C\frac{(2\lambda_1)^n}{\exp\left(\frac{|\mathcal{I}(J)|}{|J|}\log(\lambda_2)\right)}.$$

This series of inequalities now implies that

$$\exp\left(\frac{|\mathcal{I}(J)|}{|J|}\log(\lambda_2)\right) \leq C(2\lambda_1)^n,$$

and so

$$\frac{|\mathcal{I}(J)|}{|J|} \leq \frac{n\log(C2\lambda_1)}{\log(\lambda_2)}.$$

We can now choose λ_2 large enough that $|\mathcal{I}| < \frac{\delta}{2}|J|$ with the same δ as before.

□

4.4.4 Proof of Theorem 4.4.2

Taking $V^{-1} = W$ in our previous theorem, we proceed to prove Theorem 4.4.2. The proof of Theorem 4.4.2 is similar to that of 4.4.1. Let $\mathcal{G}_{\lambda,1}(J)$ denote the dyadic subintervals of J that are not contained in any interval in $\mathcal{I}'_{\lambda,1}(J)\cup\mathcal{I}_{\lambda,1}(J)$. Then define iteratively

$$\mathcal{G}_{\lambda,k}(J) = \cup_{I\in\{\mathcal{I}'_{\lambda,k-1}(J)\cup\mathcal{I}_{\lambda,k-1}(J)\}}\mathcal{G}_{\lambda,1}(I).$$

The intervals in $\{\mathcal{G}_{\lambda,k}(J)\}$ form a decomposition of the dyadic subintervals of J. Let $f\in L^2(\mathbb{R},\mathbb{C}^n)$ have finite Haar expansion and assume without loss that the support of f is $[0,1]$. We will write $\mathcal{H}_k$ for $\mathcal{I}'_{\lambda,k}([0,1])\cup\mathcal{I}_{\lambda,k}([0,1])$ and $\mathcal{G}_k$ for $\mathcal{G}_{\lambda,k}([0,1])$. Define

$$\triangle_j f = \sum_{K\in\mathcal{G}_j} h_K f_K,$$

and

$$R_j f = M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1} \sum_{K \in \mathcal{G}_j} h_K f_K = V^{-\frac{1}{2}} \sum_{K \in \mathcal{G}_j} \left\langle V^{-1} \right\rangle_K^{-\frac{1}{2}} h_K f_K.$$

We can see that

$$\sum_{j=1}^{\infty} \triangle_j f = f,$$

and that

$$\sum_{j=1}^{\infty} R_j f = M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1} f.$$

As with the square function involving two weights, we will show that $R = \sum_{j=1}^{\infty} R_j$ is bounded using Cotlar's Lemma. Firstly as R_j is zero outside of intervals in $\mathcal{H}_{j-1}$,

$$||R_j f||_{L^2}^2 = \int_{\cup \mathcal{H}_{j-1}} ||R_j f||_{\mathbb{C}^n}^2 dx = \int_{\cup \mathcal{H}_{j-1} \backslash \mathcal{H}_j} ||R_j f||_{\mathbb{C}^n}^2 dx + \int_{\cup \mathcal{H}_j} ||R_j f||_{\mathbb{C}^n}^2 dx.$$

We estimate each term in this sum separately.

$$\int_{\cup \mathcal{H}_{j-1} \backslash \mathcal{H}_j} ||R_j f||_{\mathbb{C}^n}^2 dx = \sum_{J \in \mathcal{H}_{j-1}} \int_{J \backslash \mathcal{H}(J)} ||R_j f||_{\mathbb{C}^n}^2 dx$$

$$= \sum_{J \in \mathcal{H}_{j-1}} \int_{J \backslash \mathcal{H}(J)} ||V^{-\frac{1}{2}}(x) \sum_{K \in \mathcal{G}(J)} \left\langle V^{-1} \right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx$$

$$\leq \sum_{J \in \mathcal{H}_{j-1}} \int_{J \backslash \mathcal{H}(J)} \sum_{K \in \mathcal{G}(J)} ||V^{-\frac{1}{2}}(x) \left\langle V^{-1} \right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx$$

$$\leq \sum_{J \in \mathcal{H}_{j-1}} \int_{J \backslash \mathcal{H}(J)} \sum_{K \in \mathcal{G}(J)} ||V^{-\frac{1}{2}}(x) \left\langle V^{-1} \right\rangle_K^{-\frac{1}{2}} ||^2 ||h_K(x) f_K||_{\mathbb{C}^n}^2 dx$$

$$= \sum_{J \in \mathcal{H}_{j-1}} \int_{J \backslash \mathcal{H}(J)} \sum_{K \in \mathcal{G}(J)} ||V^{-\frac{1}{2}}(x) \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} ||^2 || \left\langle V^{-1} \right\rangle_J^{\frac{1}{2}} \left\langle V^{-1} \right\rangle_K^{-\frac{1}{2}} ||^2 ||h_K(x) f_K||_{\mathbb{C}^n}^2 dx$$

$$\leq \sum_{J \in \mathcal{H}_{j-1}} \int_{J \backslash \mathcal{H}(J)} \sum_{K \in \mathcal{G}(J)} \lambda_1 || \left\langle V^{-1} \right\rangle_J^{\frac{1}{2}} \left\langle V^{-1} \right\rangle_K^{-\frac{1}{2}} ||^2 ||h_K(x) f_K||_{\mathbb{C}^n}^2 dx. \tag{4.4.7}$$

We know that almost everywhere on $J \backslash (\mathcal{I}(J) \cup \mathcal{I}'(J))$ we have $\left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} V^{-1}(x) \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \leq \lambda_1$. We also know that all the eigenvalues of $\left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}} \left\langle V^{-1} \right\rangle_K \left\langle V^{-1} \right\rangle_J^{-\frac{1}{2}}$ will be greater than $\frac{1}{\lambda_2}$, and so all of the eigenvalues of $\left\langle V^{-1} \right\rangle_J^{\frac{1}{2}} \left\langle V^{-1} \right\rangle_K^{-1} \left\langle V^{-1} \right\rangle_J^{\frac{1}{2}}$ will be less than or equal to

λ_2. It follows that $|| \left\langle V^{-1}\right\rangle_J^{\frac{1}{2}} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} || \leq \lambda_2^{\frac{1}{2}}$. And so our series of inequalities continues with (4.4.7) being less than or equal to

$$\sum_{J\in\mathcal{H}_{j-1}} \int_{J\setminus\mathcal{H}(J)} \sum_{K\in\mathcal{G}(J)} \lambda_1\lambda_2 ||h_K(x)f_K||^2_{\mathbb{C}^n} dx \leq \lambda_1\lambda_2 ||\triangle_j f||^2_{L^2}.$$

Now we consider

$$\int_{\cup\mathcal{H}_j} ||R_j f||^2_{\mathbb{C}^n} dx = \sum_{J\in\mathcal{H}_{j-i}} \sum_{I\in\mathcal{H}(J)} \int_I ||V(x)^{-\frac{1}{2}} \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K(x)||^2_{\mathbb{C}^n}.$$

As h_K is constant on $I \in \mathcal{H}(J)$ for each $K \in \mathcal{G}$, this is equal to

$$\sum_{J\in\mathcal{H}_{j-i}} \sum_{I\in\mathcal{H}(J)} \int_I \left\langle V(x)^{-1} \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K(x), \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K(x) \right\rangle_{\mathbb{C}^n}$$

$$= \sum_{J\in\mathcal{H}_{j-i}} \sum_{I\in\mathcal{H}(J)} \left\langle \int_I V(x)^{-1} dx \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K, \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K \right\rangle_{\mathbb{C}^n}$$

$$= \sum_{J\in\mathcal{H}_{j-i}} \sum_{I\in\mathcal{H}(J)} \int_I || \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K||^2_{\mathbb{C}^n}$$

$$\leq \sum_{J\in\mathcal{H}_{j-i}} \sum_{I\in\mathcal{H}(J)} \int_I || \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} \left\langle V^{-1}\right\rangle_J^{-\frac{1}{2}} ||^2 || \left\langle V^{-1}\right\rangle_J^{\frac{1}{2}} \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} f_K h_K||^2_{\mathbb{C}^n} \leq 2\lambda_1\lambda_2 ||\triangle_j f||^2_{L^2}.$$

The last inequality holds, again, due to the fact that all the eigenvalues of $\left\langle V^{-1}\right\rangle_J^{-\frac{1}{2}} \left\langle V^{-1}\right\rangle_K \left\langle V^{-1}\right\rangle_J^{-\frac{1}{2}}$ will be greater than $\frac{1}{\lambda_2}$, and so all of the eigenvalues of $\left\langle V^{-1}\right\rangle_J^{\frac{1}{2}} \left\langle V^{-1}\right\rangle_K^{-1} \left\langle V^{-1}\right\rangle_J^{\frac{1}{2}}$ will be less than or equal to λ_2. We also use the fact that I is a maximal dyadic subinterval of J to see that $\int_I || \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} \left\langle V^{-1}\right\rangle_J^{-\frac{1}{2}} ||^2 \leq 2\lambda_1$.

We have shown that there exists a constant C such that $||R_j f||^2_{L^2} \leq C||\triangle_j f||^2_{L^2}$. Now in order to deduce boundedness of R from Cotlar's Lemma, we need to show that there exists constants C' and $0 < d < 1$ such that for $k > j$,

$$\int_{\cup\mathcal{H}_{k-1}} ||R_j f||^2 \leq C' d^{k-j} ||\triangle_j f||^2.$$

Let us proceed to estimate

$$\int_{\cup\mathcal{H}_{k-1}} ||R_j f||^2 = \sum_{J\in\mathcal{H}_j} \sum_{I\in\mathcal{H}_{k-j-1}(J)} \int_I ||V^{-\frac{1}{2}}(x) \sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx$$

$$= \sum_{J\in\mathcal{H}_{j-1}} \sum_{I\in\mathcal{H}(J)} \int_{\mathcal{H}_{k-j-1}(I)} ||V^{-\frac{1}{2}}(x) \sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2 dx.$$

We now apply Hölder's inequality with p such that $2p$ is the r from our reverse Hölder inequality on V^{-1}. The expression above is then estimated by

$$\sum_{J\in\mathcal{H}_{j-1}} \sum_{I\in\mathcal{H}(J)} \left(\int_{\mathcal{H}_{k-j-1}(I)} ||V^{-\frac{1}{2}}(x) \sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^{2p} dx \right)^{\frac{1}{p}} |\mathcal{H}_{k-j-1}(I)|^{\frac{1}{q}}.$$

As our stopping time is decaying, this will be less than or equal to

$$\sum_{J\in\mathcal{H}_{j-1}} \sum_{I\in\mathcal{H}(J)} \left(\int_{\mathcal{H}_{k-j-1}(I)} ||V^{-\frac{1}{2}}(x) \sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^{2p} dx \right)^{\frac{1}{p}} \delta^{\frac{k-j-1}{q}} |I|^{\frac{1}{q}},$$

where $0 < \delta < 1$. Noting that $\sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K$ is constant on $\mathcal{H}_{k-j-1}(I)$, we can apply the reverse Hölder inequality for V^{-1} with vector $\sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K$ to see that our expression is less than or equal to

$$\sum_{J\in\mathcal{H}_{j-1}} \sum_{I\in\mathcal{H}(J)} || \left\langle V^{-1}\right\rangle_I^{\frac{1}{2}} \sum_{L\in\mathcal{H}_{j-1}} \sum_{K\in\mathcal{G}(L)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{L^2}^{2p} (C'')^{\frac{1}{p}} |I|^{\frac{1}{p}} \delta^{\frac{k-j-1}{q}} |I|^{\frac{1}{q}}$$

$$= \delta^{\frac{k-j-1}{q}} (C'')^{\frac{1}{p}} \sum_{J\in\mathcal{H}_{j-1}} \int_{\cup\mathcal{H}(J)} ||V^{-\frac{1}{2}}(x) \sum_{K\in\mathcal{G}(J)} \left\langle V^{-1}\right\rangle_K^{-\frac{1}{2}} h_K(x) f_K||_{\mathbb{C}^n}^2$$

$$= \delta^{\frac{k-j-1}{q}} (C'')^{\frac{1}{p}} \sum_{J\in\mathcal{H}_{j-1}} \int_{\cup\mathcal{H}(J)} ||R_j f||^2,$$

which by our previous estimate we know is less than or equal to

$$\delta^{\frac{k-j-1}{q}} (C'')^{\frac{1}{p}} C ||\triangle_j f||^2.$$

As before, we now have the required estimates to use Cotlar's lemma to prove the boundedness of $R = M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$.

Note that we can also prove Corollary 4.3.2 using only Theorem 4.4.2 rather than both Theorem 4.4.1 and Theorem 4.4.2:

Proof of Corollary 4.3.2. $M_V^{-\frac{1}{2}} T_\sigma M_U^{\frac{1}{2}}$ can be written as

$$M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1} D_{V^{-1}} T_\sigma D_U D_U^{-1} M_U^{\frac{1}{2}}.$$

Note that T_σ commutes with $D_{V^{-1}}$ and we can estimate the norm as follows

$$||M_V^{-\frac{1}{2}} T_\sigma M_U^{\frac{1}{2}}|| = ||M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1} D_{V^{-1}} T_\sigma D_U D_U^{-1} M_U^{\frac{1}{2}}||$$

$$\leq ||M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}|| ||T_\sigma|| ||D_{V^{-1}} D_U|| ||D_U^{-1} M_U^{\frac{1}{2}}||.$$

We need conditions on U and V that imply that the operators $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$, $D_{V^{-1}} D_U$ and $D_U^{-1} M_U^{\frac{1}{2}}$ are bounded. Theorem 4.4.2 immediately gives us that $D_U^{-1} M_U^{\frac{1}{2}}$ is bounded. This theorem also applies to $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$ if we note that its adjoint is $D_{V^{-1}}^{-1} M_V^{-\frac{1}{2}}$. All we need to show now is that under the hypothesis $D_{V^{-1}} D_U$ is a bounded operator. This follows from the joint A_2 condition. $\square$

4.5 Application to the Hilbert Transform

As well as showing that the martingale transforms are uniformly bounded under the conditions of the two main theorems, we can also show that the dyadic shift, Ш from Definition 1.9.1, will be bounded and hence the Hilbert transform is bounded by way of S. Petermichl's averaging techniques, see [27] and [28].

Definition 4.5.1. Define the operator D_V^+ as

$$D_V^+ : f = \sum_{I \in \mathcal{D}} f_I h_I \mapsto \sum_{I \in \mathcal{D}} \langle V \rangle_{I_+}^{\frac{1}{2}} f_I h_I,$$

and the operator D_V^- as

$$D_V^- : f = \sum_{I \in \mathcal{D}} f_I h_I \mapsto \sum_{I \in \mathcal{D}} \langle V \rangle_{I_-}^{\frac{1}{2}} f_I h_I,$$

for $f \in L^2(\mathbb{C}^n)$ with finite Haar expansion.

If we split the shift operator into a sum of two operators, each of which is bounded,

$$Ш f = (Ш_1 + Ш_2) f = \sum_{I \in \mathcal{D}} \left(\langle f, h_{I_+} \rangle h_I \right) - \sum_{I \in \mathcal{D}} \left(\langle f, h_{I_-} \rangle h_I \right),$$

we can then check that $D_{V^{-1}}^+ Ш_1 D_{V^{-1}}^{-1} = Ш_1$ and $D_{V^{-1}}^- Ш_2 D_{V^{-1}}^{-1} = Ш_2$. As before, we can estimate $||M_U^{\frac{1}{2}} Ш M_{V^{-1}}^{\frac{1}{2}}||$ as follows:

$$||M_U^{\frac{1}{2}} (Ш_1 + Ш_2) M_{V^{-1}}^{\frac{1}{2}}|| = ||M_U^{\frac{1}{2}} \left(D_{V^{-1}}^+ Ш_1 D_{V^1}^{-1} + D_{V^{-1}}^- Ш_2 D_{V^{-1}}^{-1} \right) M_{V^{-1}}^{\frac{1}{2}}||$$

$$\leq \left(||M_U^{\frac{1}{2}} D_{V^{-1}}^{+}|| \, ||Ш_1|| + ||M_U^{\frac{1}{2}} D_{V^{-1}}^{-}|| \, ||Ш_2|| \right) ||D_{V^{-1}}^{-1} M_{V^{-1}}^{\frac{1}{2}}||.$$

We have already dealt with the boundedness of the third operator, and it is known that $Ш_1$ and $Ш_2$ are bounded on unweighted L^2. This leaves the operators $M_U^{\frac{1}{2}} D_{V^{-1}}^{+}$ and $M_U^{\frac{1}{2}} D_{V^{-1}}^{-}$.

$$||(D_{V^{-1}}^{+}) M_U^{\frac{1}{2}} f||^2 = \left\langle M_U^{\frac{1}{2}} (D_{V^{-1}}^{+})^2 M_U^{\frac{1}{2}} f, f \right\rangle$$

$$= \left\langle (D_{V^{-1}}^{+})^2 M_U^{\frac{1}{2}} f, M_U^{\frac{1}{2}} f \right\rangle = \left\langle \sum_{I \in \mathcal{D}} \langle V^{-1} \rangle_{I_+} (U^{\frac{1}{2}} f)_I h_I, M_U^{\frac{1}{2}} f \right\rangle$$

$$= \sum_{I \in \mathcal{D}} \left\langle \langle V^{-1} \rangle_{I_+} (U^{\frac{1}{2}} f)_I h_I, M_U^{\frac{1}{2}} f \right\rangle = \sum_{I \in \mathcal{D}} \frac{1}{|I_+|} \int_{I_+} \left\langle V^{-1}(x) (U^{\frac{1}{2}} f)_I h_I, M_U^{\frac{1}{2}} f \right\rangle dx$$

$$= \sum_{I \in \mathcal{D}} \frac{1}{|I_+|} \int_{I_+} \left\langle V^{-1}(x) (U^{\frac{1}{2}} f)_I h_I, (U^{\frac{1}{2}} f)_I h_I \right\rangle dx \leq \sum_{I \in \mathcal{D}} \frac{1}{|I_+|} \int_{I} \left\langle V^{-1}(x) (U^{\frac{1}{2}} f)_I h_I, U^{\frac{1}{2}} f \right\rangle dx$$

$$= \sum_{I \in \mathcal{D}} \frac{2}{|I|} \int_{I} \left\langle V^{-1}(x) (U^{\frac{1}{2}} f)_I h_I, U^{\frac{1}{2}} f \right\rangle dx = 2 \left\langle M_U^{\frac{1}{2}} (D_{V^{-1}})^2 M_U^{\frac{1}{2}} f, f \right\rangle.$$

The inequality is true because we are integrating a positive function, $\left\langle V^{-1}(x) (U^{\frac{1}{2}} f)_I h_I, (U^{\frac{1}{2}} f)_I h_I \right\rangle = \left\langle V^{-\frac{1}{2}}(x) (U^{\frac{1}{2}} f)_I h_I, V^{-\frac{1}{2}}(x) (U^{\frac{1}{2}} f)_I h_I \right\rangle$, over a larger interval. The second last equality is due to the fact that $|I_+| = \frac{1}{2}|I|$. The boundedness of $M_U^{\frac{1}{2}} D_{V^{-1}}^{+}$ then follows from our previous bounding of $M_U^{\frac{1}{2}} D_{V^{-1}}$ and taking adjoints where appropriate. For $M_U^{\frac{1}{2}} D_{V^{-1}}^{-}$, the proof is similar.

Instead of this canonical dyadic grid we can define the shift operator, $Ш^{\beta,r}$ from Definition 1.9.1, on the grid $\mathbb{D}_{r,\beta} = \left\{ r 2^m \left([0,1) + l + \sum_{n<m} 2^{i-n} \beta_i \right) \right\}_{l,m \in \mathbb{Z}}$. The shift operators defined with respect to these dyadic grids will be bounded $L^2(V) \to L^2(U)$ given the joint A_2 condition is satisfied, U satisfies the reverse Hölder condition and V^{-1} the $A_{2,0}$ condition, all on this new lattice. The resulting estimate for the norm will be independent of the lattice.

Assuming the joint A_2 condition, that U satisfies the reverse Hölder condition and V the $A_{2,0}$ condition, all on arbitrary intervals, allows us to estimate the norm of the Hilbert transform in terms of these translated and dilated Haar shifts using the results from [27] and [28].

Theorem 4.5.2. *Let U and V be matrix weights satisfying the joint (U,V) A_2 pair condition:*

$$\langle V^{-1} \rangle_I^{\frac{1}{2}} \langle U \rangle_I \langle V^{-1} \rangle_I^{\frac{1}{2}} < C$$

for all intervals I, where C is a constant multiple of the identity. If $V^{-1} \in A_{2,0}$ and U satisfies the matrix reverse Hölder inequality, then the Hilbert transform is bounded from $L^2(V)$ to $L^2(U)$.

Proof.

$$\left|\left\langle M_U^{\frac{1}{2}} H M_V^{-\frac{1}{2}} f, g\right\rangle\right| = C\left|\left\langle M_U^{\frac{1}{2}} \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 Ш^{\beta,r} M_V^{-\frac{1}{2}} f \frac{dr}{r} d\mathbb{P}(\beta), g\right\rangle\right|$$

$$= C\left|\int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 \left\langle M_U^{\frac{1}{2}} Ш^{\beta,r} M_V^{-\frac{1}{2}} f, g\right\rangle \frac{dr}{r} d\mathbb{P}(\beta)\right|$$

$$\leq C \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 \left|\left\langle M_U^{\frac{1}{2}} Ш^{\beta,r} M_V^{-\frac{1}{2}} f, g\right\rangle\right| \frac{dr}{r} d\mathbb{P}(\beta)$$

$$\leq C\widetilde{C} \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 |\langle f, g\rangle| \frac{dr}{r} d\mathbb{P}(\beta) \leq CC^* ||f||_{L^2} ||g||_{L^2},$$

where C is the constant appearing in Theorem 1.9.3 and $\widetilde{C}$ is the uniform operator norm of the shift operators. □

The idea for adapting our main argument to the case of the dyadic shift can be applied to a more general class of operators, *band operators.*

4.6 Application to band operators and certain singular integral operators

Definition 4.6.1. A band operator T is a bounded operator on $L^2(\mathbb{R})$ such that there exists an integer $r > 0$ for which $\langle Th_I, h_J\rangle = 0$ for all Haar functions h_I and h_J, where J is at least a distance of r away from I. By distance we mean tree distance between dyadic intervals where the tree is formed by connecting each interval with its parent and children intervals. The distance between each parent and child being 1. See figures 4.1 and 4.2 for an illustration of this.

One crucial fact is that, for each r there are only a finite number of Haar basis elements $h_{\tilde{I}}$ less than tree distance r from h_I. Suppose there are m Haar basis elements less than r away from each h_I and we label these basis elements h_{I_i} for $i = 1..m$. Then our band operator T will be of the form

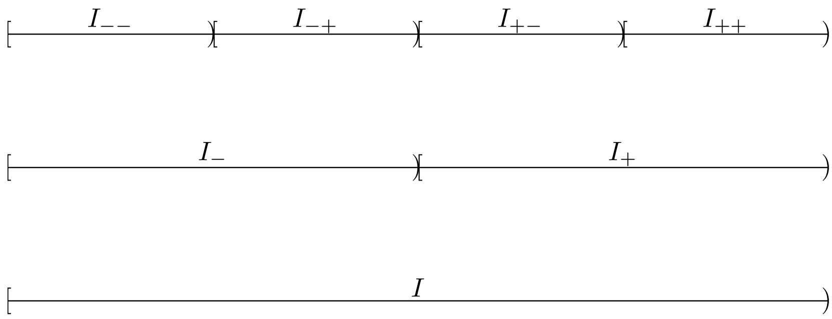

Figure 4.1: A dyadic interval I together with first and second generation subintervals.

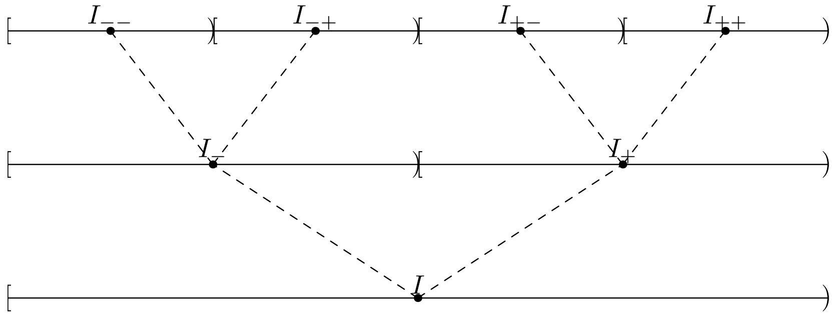

Figure 4.2: The tree formed by connecting dyadic intervals to their parents and children.

$$f \mapsto \sum_{I \in \mathcal{D}} \sum_{i=1}^{m} \phi(I, I_i) \langle f, h_I \rangle h_{I_i},$$

where ϕ is a function from $\mathcal{D} \bigoplus \mathcal{D}$ to $\mathbb{C}$.

Lemma 4.6.2. *If we have a band operator, T, written in the form*

$$f \mapsto \sum_{I \in \mathcal{D}} \sum_{i=1}^{m} \phi(I, I_i) \langle f, h_I \rangle h_{I_i},$$

then the function $\phi : \mathcal{D} \bigoplus \mathcal{D} \to \mathbb{C}$ is bounded.

Proof. Suppose that ϕ is unbounded. As T is a bounded operator we can choose I and I_i such that $|\phi(I, I_i)| > ||T||$. Then we can see that

$$||Th_I|| = ||\sum_{I \in \mathcal{D}} \sum_{i=1}^{m} \phi(I, I_i) \langle h_I, h_I \rangle h_{I_i}|| = ||\sum_{i=1}^{m} \phi(I, I_i) h_{I_i}|| = \left(\sum_{i=1}^{m} |\phi(I, I_i)|^2 \right)^{\frac{1}{2}} > ||T||,$$

condradicting our hypothesis that T is bounded. □

Theorem 4.6.3. *Let (U, V) be a matrix A_2 pair. If $V^{-1} \in A_{2,0}$ and U satisfies the dyadic matrix reverse Hölder inequality, then any band operator T is bounded from $L^2(V)$*

to $L^2(U)$. If r is the maximum distance associated to the band operator then the bound will depend only on r, the $L^2 \to L^2$ norm of the operator and the $A_{2,0}$ constant and reverse Hölder constants associated to the weights.

Proof. Again we note that

$$Tf = \sum_{I \in \mathcal{D}} \sum_{i=1}^{m} \phi(I, I_i) \langle f, h_I \rangle h_{I_i},$$

where ϕ is a function from $\mathcal{D} \bigoplus \mathcal{D}$ to $\mathbb{C}$. The intervals I and I_i will always share an ancestor less than r generations away for each $i = 1..m$. In the case that I_i is a descendant of I then I will be the common ancestor. In the case where I_i is an ancestor of I then I_i will be the common ancestor. It is also possible to be in a situation where neither of these are true but the intervals still share a common ancestor no more than distance r away.

We can split T into a sum of m bounded operators

$$T = \sum_{i=1}^{m} T_i,$$

where T_i is the operator

$$f \mapsto \sum_{I \in \mathcal{D}} \phi(I_i, I) \langle f, h_{I_i} \rangle h_I.$$

This sum is constructed so that for each summand T_i and Haar basis element h_I there is exactly one Haar coefficient, $\langle f, h_i \rangle$, being mapped to h_I. Due to the nature of the band operator there are at most m Haar coefficients being mapped to each basis element, and thus it is possible to decompose T into a finite sum of these operators.

We proceed to estimate $||M_U^{\frac{1}{2}} T M_{V^{-1}}^{\frac{1}{2}}||$. Note that

$$TD_{V^{-1}} = \left(\sum_{i=1}^{m} T_i \right) D_{V^{-1}} = \sum_{i=1}^{m} D_{V^{-1}}^{i} T_i,$$

where $D_{V^{-1}}^{i}$ is the operator

$$f \mapsto \sum_{I \in \mathcal{D}} \langle V^{-1} \rangle_{I_i}^{\frac{1}{2}} f_I h_I.$$

So

$$||M_U^{\frac{1}{2}} T M_{V^{-1}}^{\frac{1}{2}}|| = ||M_U^{\frac{1}{2}} T D_{V^{-1}} D_{V^{-1}}^{-1} M_{V^{-1}}^{\frac{1}{2}}|| = ||M_U^{\frac{1}{2}} \left(\sum_{i=1}^{m} D_{V^{-1}}^{i} T_i \right) D_{V^{-1}}^{-1} M_{V^{-1}}^{\frac{1}{2}}||$$

$$\leq \left(\sum_{i=1}^{m} ||M_U^{\frac{1}{2}} D_{V^{-1}}^{i} T_i|| \right) ||D_{V^{-1}}^{-1} M_{V^{-1}}^{\frac{1}{2}}||$$

$$\leq \left(\sum_{i=1}^{m} ||M_U^{\frac{1}{2}} D_{V^{-1}}^i|| ||T_i||\right) ||D_{V^{-1}}^{-1} M_{V^{-1}}^{\frac{1}{2}}||.$$

We know that each T_i is bounded and we have already dealt with the boundedness of $D_{V^{-1}}^{-1} M_{V^{-1}}^{\frac{1}{2}}$. So it remains to bound each $M_U^{\frac{1}{2}} D_{V^{-1}}^i$.

For any $f \in L^2$,

$$||M_U^{\frac{1}{2}}(D_{V^{-1}}^i)f||_{L^2}^2 = \left\langle M_U^{\frac{1}{2}}(D_{V^{-1}}^i)^2 M_U^{\frac{1}{2}} f, f\right\rangle_{L^2} = \left\langle (D_{V^{-1}}^i)^2 M_U^{\frac{1}{2}} f, M_U^{\frac{1}{2}} f\right\rangle_{L^2}$$

$$= \left\langle \sum_{I\in\mathcal{D}} \left\langle V^{-1}\right\rangle_{I_i} (U^{\frac{1}{2}} f)_I h_I, (U^{\frac{1}{2}} f)_I h_I \right\rangle_{L^2} = \sum_{I\in\mathcal{D}} \left\langle \left\langle V^{-1}\right\rangle_{I_i} (U^{\frac{1}{2}} f)_I h_I, (U^{\frac{1}{2}} f)_I h_I \right\rangle_{L^2}$$

$$= \sum_{I\in\mathcal{D}} \frac{1}{|I_i|} \int_{I_i} \int_I \left\langle V^{-1}(x)(U^{\frac{1}{2}} f)_I h_I(y), (U^{\frac{1}{2}} f)_I h_I(y)\right\rangle_{\mathbb{C}^n} dydx$$

$$\leq \sum_{I\in\mathcal{D}} \frac{2^r}{|I'|} \int_{I'} \int_I \left\langle V^{-1}(x)(U^{\frac{1}{2}} f)_I h_I(y), (U^{\frac{1}{2}} f)_I h_I(y)\right\rangle_{\mathbb{C}^n} dydx,$$

where I' is the common ancestor of I and I_i. This is true because each term

$$\left\langle V^{-1}(x)(U^{\frac{1}{2}} f)_I h_I, (U^{\frac{1}{2}} f)_I h_I\right\rangle_{\mathbb{C}^n} = \left\langle V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}} f)_I h_I, V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}} f)_I h_I\right\rangle_{\mathbb{C}^n}$$

is positive.

We have seen before that if a matrix weight U satisfies the dyadic $A_{2,0}$ condition, then for any vector γ the scalar weight $||U^{\frac{1}{2}}\gamma||^2$ will satisfy the scalar dyadic A_∞ condition. So, if we have a dyadic interval I and a dyadic interval J contained in I such that the tree distance between these two is less than r, i.e. $|I| \leq 2^r|J|$, then one of the standard properties of A_∞, see [32] page 196, tells us that

$$\beta \int_I ||U^{\frac{1}{2}}\gamma||^2 \leq \int_J ||U^{\frac{1}{2}}\gamma||^2$$

for some $0 < \beta < 1$ bounded away from 0, with the bound dependent only on r and the A_∞ constant.

Using our hypothesis that $V^{-1} \in A_{2,0}$, we can see that

$$\sum_{I\in\mathcal{D}} \frac{2^r}{|I'|} \int_{I'} \int_I \left\langle V^{-1}(x)(U^{\frac{1}{2}} f)_I h_I(y), (U^{\frac{1}{2}} f)_I h_I(y)\right\rangle_{\mathbb{C}^n} dydx$$

$$= \sum_{I\in\mathcal{D}} \frac{2^r}{|I'|} \int_{I'} \int_I \left\langle V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}} f)_I h_I(y), V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}} f)_I h_I(y)\right\rangle_{\mathbb{C}^n} dydx$$

$$= \sum_{I\in\mathcal{D}} \frac{2^r}{|I'|} \int_{I'} ||V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}} f)_I||_{\mathbb{C}^n}^2 \int_I |h_I(y)|^2 dydx$$

$$= \sum_{I\in\mathcal{D}} \frac{2^r}{|I'|} \int_{I'} ||V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}}f)_I||^2_{\mathbb{C}^n} dx \leq \sum_{I\in\mathcal{D}} \frac{2^r}{\beta|I|} \int_I ||V^{-\frac{1}{2}}(x)(U^{\frac{1}{2}}f)_I||^2_{\mathbb{C}^n} dx$$

$$= \sum_{I\in\mathcal{D}} \frac{2^r}{\beta} || \left\{ \frac{1}{|I|} \int_I V^{-1}(x)dx \right\}^{\frac{1}{2}} (U^{\frac{1}{2}}f)_I||^2_{\mathbb{C}^n} = \frac{2^r}{\beta} ||D_{V^{-1}} M_U^{\frac{1}{2}} f||^2.$$

This reduces the estimate of each $D^i_{V^{-1}} M_U^{\frac{1}{2}}$ to $D_{V^{-1}} M_U^{\frac{1}{2}}$, which was dealt with in Theorem 4.4.1. $\square$

If K is a function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$ that is twice differentiable, and such that the function $x^3K(x)$ is almost everywhere bounded, and the limit as $x \to \infty$ of both $K(x)$ and the first derivative $K'(x)$ are 0, then the following theorem due to Vagharshakyan allows us to apply our hypothesis to singular integral operators of convolution type with such kernels K. Vagharshakyan's theorem models singular integral operators with such kernels in terms of translations and dilations of band operators.

Theorem 4.6.4 (Vagharshakyan). *If T is a singular integral operator of convolution type with kernel K as defined above, then T is a positive multiple of the following operator*

$$f \mapsto \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 B^{\beta,r} f \frac{dr}{r} d\mathbb{P}(\beta),$$

where $B^{\beta,r}$ is a band operator defined in terms of the dyadic grid $\mathbb{D}_{\beta,r}$ exactly as they are defined for the canonical dyadic grid.

Theorem 4.6.5. *Let (U, V) be an A_2 pair, i.e.*

$$\langle V^{-1}\rangle_I^{\frac{1}{2}} \langle U\rangle_I \langle V^{-1}\rangle_I^{\frac{1}{2}} < C$$

for all intervals I, where C is a constant multiple of the identity. If $V^{-1} \in A_{2,0}$ and U satisfies the matrix reverse Hölder inequality, then the singular integral operator of convolution type with kernel K is bounded from $L^2(V)$ to $L^2(U)$.

Proof.

$$\left|\left\langle M_U^{\frac{1}{2}} T M_V^{-\frac{1}{2}} f, g\right\rangle\right| = \tilde{C} \left|\left\langle M_U^{\frac{1}{2}} \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 B^{\beta,r} M_V^{-\frac{1}{2}} f \frac{dr}{r} d\mathbb{P}(\beta), g\right\rangle\right|$$

$$= \tilde{C} \left| \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 \left\langle M_U^{\frac{1}{2}} B^{\beta,r} M_V^{-\frac{1}{2}} f, g\right\rangle \frac{dr}{r} d\mathbb{P}(\beta) \right|$$

$$\leq \tilde{C} \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 \left|\left\langle M_U^{\frac{1}{2}} B^{\beta,r} M_V^{-\frac{1}{2}} f, g\right\rangle\right| \frac{dr}{r} d\mathbb{P}(\beta)$$

$$\leq \tilde{C}C^* \int_{\{0,1\}^{\mathbb{Z}}} \int_1^2 ||f||_2||g||_2 \frac{dr}{r} d\mathbb{P}(\beta) \leq \tilde{C}C^*||f||||g||,$$

where $\tilde{C}$ is the constant multiple of the singular integral operator corresponding to the average of the band operators and C^* is a bound on the operator norms of the band operators. Note by uniform norm we mean that, a particular band operator, then defined with respect to different dyadic grids, will have the same operator norm. $\square$

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